MULTIPLE PERSPECTIVES ON LANGUAGE AND MATHEMATICS: INTRODUCTION AND POST-SCRIPT

Candia Morgan, Institute of Education, University of London
Konstantinos Tatsis, University of the Aegean
Hana Moraová, Charles University in Prague, Faculty of Philosophy & Arts
Jarmila Novotná, Charles University in Prague, Faculty of Education
Margarida César, Universidade de Lisboa, Centro de Investigação em Educação FCUL
Birgit Brandt, Johann Wolfgang Goethe-Universität, Frankfurt a. Main
Elmar Cohors-Fresenborg & Christa Kaune, Institut für Kognitive Mathematik, Universität Osnabrück, D-49069 Osnabrück

INTRODUCTION

Candia Morgan

The papers presented and the discussions of the Working Group on Language and Mathematics at CERME5 were marked by, on the one hand, diversity in the orientations and research foci of the various participants and, on the other hand, an interest in establishing dialogue and engaging with each other's questions, data and analyses. A concrete outcome of the opportunity to meet provided by the conference was an agreement to do some 'homework' resulting in new analyses, from several perspectives, of data presented in two of the conference papers. This introduction to the papers of the Working Group starts with an overview of the major themes emerging from the papers and from our discussions. It then presents the outcomes of the 'homework'.

In recent years, there has been increased recognition of the importance of language, not just as a means of communication but as a means by which we make sense of, or even construct, the world. This has led to a widening of the community of those within mathematics education who see language as a significant focus for their research and a consequent widening of the orientations of those choosing to participate in the Working Group on Language and Mathematics. Two main research orientations can be identified: study of the nature of language and its use in doing and learning mathematics and study of other issues, using language as a tool for addressing them. Within these two broad orientations there is also considerable diversity. For example, in studying the nature of language used in mathematics, choices must be made about the level of granularity at which the language is to be studied. The focus may vary from the nature and functioning of individual signs or small sets of signs, as seen in the work of, among others, Steinbring & Nührenbörger, Bloch and Farrugia, to consideration at a much more holistic level of the nature and function of writing in mathematical practices, as in Misfeldt's study of professional mathematicians or Stamou & Chronaki's analysis of the discourse of a magazine for

school students. In between these two may be found studies of the functioning of language within spoken or written discourse and its contribution to the construction of mathematical meaning (e.g. Boero & Consogno). Where language is studied as a vehicle to address other issues of primary interest, there is perhaps even more space for diversity. Among the papers presented here, we encounter studies of student attitudes and beliefs (Perkkila & Aarnos), of the development of socio-mathematical norms (Edwards) and their influence on problem solving (Tatsis), and of assessment (Björklund Boistrup). A concern with the nature of learning environments that may facilitate learning is apparent in studies of linguistic activity and interaction in classrooms (e.g. Fetzer, Brandt) and in Ascione & Mellone's experimental study. The development of methodological approaches to the analysis of linguistic data were also offered in the papers by Koichu (for studying cognitive processes in problem solving) and Cohors-Fresenborg & Kaune (for identifying and categorising meta-cognitive activities).

The important role that linguistic activity (and that involving other sign systems) plays in the construction of mathematical meaning is widely recognised and many of the contributing authors present analyses of written texts and verbal interactions that contribute to our understanding of aspects of this. The social and collaborative aspect is of particular interest as we focus on interactions and learning that takes place in 'natural' classrooms rather than in laboratory or interview settings. At the same time, it must be recognised that participants in interactions do not always successfully collaborate to construct coherent meanings. Several contributions identify problems in communication that may constitute 'obstacles' to learning (Petrová & Novotná, Roubíček, Slezáková & Swoboda), while Farrugia identifies 'clarity' in teacher's speech as one of the keys to student learning and attempts an analysis of its characteristics. Students' acquisition of mathematical language is clearly an important aspect of communication that may support learning, yet, as Meaney demonstrates in her analysis of the role of authority in communications between children, teachers and parents, developing competence in use of the mathematics register itself raises difficult issues. The relationship between linguistic activity, linguistic competence and mathematical learning and competence is complex and as yet unresolved. When we consider students' mathematical competence, to what extent is their linguistic competence a part of this? And conversely, when we analyse interaction in a classroom or in an interview, how is this interaction affected by the mathematical aspects of the context?

An interesting development in the past few years has been the increasing attention to alternative, non-linguistic sign systems. We noted in the introduction to the proceedings of the Working Group at CERME4 (Morgan, Ferrari, Johnsen Høines, Duval, 2006) the importance of recognising and analysing the nature and roles of algebraic notation and geometric diagrams as well as 'natural' language. While work on these aspects continues, several contributions to CERME5 go further to consider other non-verbal sign systems such as gesture, body language and gaze. Björklund

Boistrup, in particular, develops a multi-modal approach to study interactions between students and teachers, taking all of these into account in addition to spoken language. As researchers develop this work with multi-modal data, the tools that are available for analysing the various modes need to be integrated and coordinated to ensure theoretical consistency.

Multi-modality is of increasing interest within mathematics education and elsewhere, especially in the context of new technologies that provide new types of signs and ways of interacting with them. It is perhaps surprising that none of the present contributions address this aspect of language, as influenced by new technology, though the papers by Back & Pratt and Pimm, Beatty & Moss consider the nature of interactions in text-based on-line environments. It may be that the multi-modal opportunities offered by technological developments are currently considered of specific interest to those concerned with the use of new technologies. As the field matures, providing more developed tools for analysis of multi-modal discourse, and as new technologies become more fully integrated into mathematical teaching and learning situations as well as into our everyday lives, it will become increasingly difficult to restrict our research focus to the more conventional and familiar mathematical sign systems.

Among other methodological issues discussed, the selection, status and treatment of data seemed particularly significant and in need of explicit clarification. Many of the papers present 'episodes' of data from classroom interactions. 'Episode', however, may be simply a fragment, perhaps chosen to illustrate a point, or it may be more 'logically' defined by its content, its interactional features or its crucial significance. The nature of episodes presented in papers is not always made explicit to the reader, yet must make a difference to the way in which the results of their analysis may be understood: as raising issues or hypotheses; as 'slices' of a developmental process that has been studied more extensively; as representative broader phenomena. It was suggested that there may be a case for complementing the use of detailed fine grained analysis of 'episodes' with larger scale quantitative approaches.

The fine grained analysis in many cases makes use of transcriptions, yet transcriptions do not necessarily provide a good representation of an episode of semiotic activity, often neglecting prosodic features as well as the coordination of linguistic with visual or physical modes. Researchers need to consider the rigour and scope of their methods of transcription. There are several well-developed sets of conventions employed by linguists for transcription. Some of these may help us to be more rigorous in representing speech but it is important to ensure that any conventions adopted are adequate to capture those features of speech considered to be significant and that the methods and conventions used match the theoretical assumptions of the research. When other modes of communication are also to be considered, the task of representing them is further complicated. Another role of technology discussed in the Working Group may provide one way of beginning to address this problem by making research tools available to us that enable us to have a

fuller view of an episode and, indeed, to ask new questions. For example, digitised video technology allows us to gather and examine more complex multi-modal data and allow us to analyse both temporal and spatial relationships between gestures, visual representations and speech. The work of Bjuland, Cestari & Borgersen begins to make use of such technology to analyse student reasoning during problem solving, as expressed through gesture and spoken language.

Perhaps as a consequence of the diversity of our backgrounds (both cultural and disciplinary), discussions were marked by simultaneous interest in the substantive research questions and findings reported by the presenters and in the methodological and theoretical issues raised. Thus, in considering the use of signs and language in meaning making, it was important to ask not only how students use signs in order to make mathematical meanings but also what linguistic and semiotic knowledge is useful to us as researchers in interpreting meaning making. By making use of different sets of theoretical constructs, different insights emerge. A shared interest in exploring these theoretical and methodological differences led to an agreement to continue working on this issue after the conference by preparing complementary analyses from different perspectives of some of the data presented. Episodes originally analysed and presented in the papers by Cohors-Fresenburg & Kaune and by Boero & Consogno were chosen for this treatment. The following sections of this paper include four brief complementary analyses by Tatsis, Moraová & Novotná, Margarida César and Birgit Brandt of an episode presented in the paper by Cohors-Fresenborg & Kaune. (For convenience, the episode in question is reproduced as an annex to this paper.) This is followed by a complementary analysis by Cohors-Fresenborg & Kaune of data from Boero & Consogno.

USING POLITENESS THEORY TO ANALYSE A CLASSROOM DISCUSSION

Konstantinos Tatsis

The linguistic analysis of classroom interactions can be used as a tool to better comprehend these interactions and then better organise the didactic approach. Cohors-Fresenborg & Kaune's analytic approach addresses the important question set by Candia Morgan during the Language and Mathematics Working Group meeting: When we analyse interaction in a classroom or in an interview, how is this interaction affected by the mathematical aspects of the context? In order to better comprehend the interactions involved in any setting (including classrooms) one needs to consider all aspects that influence in one way or another what is said and what is done. The most important aspect that affects people's behaviour is "face", i.e. "the positive social value a person effectively claims for himself by the line others assume he has taken during a particular contact" (Goffman, 1972, p. 5). Face is further categorised into positive and negative: positive face is related to a person's need for social approval, whereas negative face is related to a person's need for freedom of action.

Each person does not only have these wants her/himself, but recognises that others have them too; moreover, s/he recognises that the satisfaction of her/his own face wants is, in part, achieved by the acknowledgement of those of others. Indeed, the nature of positive face wants is such that they can only be satisfied by the attitudes of others. These views are in the core of "politeness theory" as expressed by Brown and Levinson (1987) and used by Rowland (2000) and will be the theoretical base for the analysis that follows.

Each verbal act can be categorised according to its effect on the speaker or the hearer's face. Some acts ("face threatening acts", or FTAs) intrinsically threaten the hearer's face. Orders and requests, for example, threaten negative face, whereas criticism and disagreement threaten positive face. Each person must avoid such acts altogether (which may be impossible for a host of reasons, including concern for her/his own face) or find ways of performing them whilst mitigating their FTA effect, i.e. making them less of a threat. Imagine, for example, that a student says something that the teacher believes to be factually incorrect; the teacher would like to correct him/her. Such an act would threaten the student's positive face; thus, the teacher has to employ a particular strategy in order to minimise the potential FTA effect.

The discussion contained in Cohors-Fresenborg & Kaune's paper is very interesting because it contains many instances of potential FTA acts, which are successfully resolved by the speakers. In 5-6 Mona supports her claim about the existence of a particular figure and the teacher, knowing that this figure does not really exists, asks for a numerical representation of it; she begins her request with the modal form "Could you please" in order to minimise the threat to Mona's negative face. Mona initially admits that it is not possible, but tries to support her view in two ways: she uses "you" on an attempt to make her claim impersonal (i.e. it is not her own inability, but a general one); then she utters that "logically it would be possible", which suggests that her claim is logical and reasonable (this utterance can refer to a possible sociomathematical norm established in the particular classroom, i.e. that a mathematical proposition is expected to be logical in order to be acceptable). In 12-13 the teacher tries to raise the others students' interest in Mona's claim; this is a FTA to Mona, that is why she immediately replies (although not asked) by using once again the impersonal "you" (14) in order to assign a general character to her claim. Suse (17-21) only repeats Mona's claim and the teacher utters "Yes" not as a sign of acceptance, but as a way to encourage more students to participate in the discussion; that is why she uses the first plural person ("let's") in her prompt. Suse (24-31) refers to Peter's and Mona's claims by using many times the impersonal "you" in order to distance herself from both of them; this is done in order to minimise the threat to her own positive face, in case they prove faulty. Mona eventually realises that her initial claim is not grounded; she begins by using the shield "Well" and gradually she admits this fact. It is interesting to observe that Mona was led to withdraw her initial claim without any interference on behalf of the teacher; this is a sign of a student who observes the sociomathematical norm of justification (for a more detailed discussion

on social and sociomathematical norms see Tatsis, this volume), which is important for a fruitful mathematical discussion.

What the above analysis demonstrates is an alternative way to look into mathematical discussions; students and teachers always adopt particular strategies to save their (or their hearer's) face. Moreover, we can use such an analysis to examine the teachers' and the students' attempts to generalise and to justify but with the minimum effect towards their own and the others' face. The educator who is aware of these strategies can better organise the discussions, and particularly his/her own verbal strategies towards smooth and productive mathematical interactions.

DISCOURSE ANALYSIS USING PRAGMATICS

Hana Moraová & Jarmila Novotná

Pragmatics is one of the three divisions of semiotics (together with semantics and syntax). It studies language from the point of view of the user, especially of the choices he/she makes, the constraints he/she encounters in using the language in social interaction, and the effects his/her use of language has on the participants of an act of communication. It focuses on language in use and relatively changing features of conversation. It studies continuous wholes (for more information see Leech 1983).

We believe that this approach is suitable for analyses of teaching episodes as it enables us to see why the participants of the communication behave in a particular manner and what the possible sources of misunderstanding may be or why individual contributions may seem "clumsy", illogical or confusing.

At the core of the analysis are major principles and their maxims, which in normal speech situation are expected not to be violated by the participants. If they are violated, it brings confusion or misunderstanding. Also, the different principles may be in opposition to each other which can cause that if one of the principles is obeyed the other violated. (E.g. the politeness principle is often in conflict with the cooperative principle – namely the quality and quantity maxims.)

In this contribution we only refer to those principles and maxims that are relevant to the particular transcript.

Analysis of the episode

Cooperative principle (for more information see Grice 1975)

- *Quality maxim* (try to make your contribution one that is true, do not say that for which you lack adequate evidence) is often violated; however this is not surprising as the conversation is from a lesson where students are expected to reason, deduce, search and will say things without having sufficient evidence for it; it happens that only after some time they realize their original assumption was wrong (Mona's assumption that a number between 0,99... and 1 exists is untrue, but progressive in the course of the lesson - l. 5).

- *Quantity maxim* (make your contribution as informative as required for the purposes of the exchange, do not make it more informative than is required): Mona's only "valuable" contribution is on line 5-6, then she keeps repeating the same idea: "logically you can imagine but you cannot write it down" (l. 8, 10, 31, 33) and thus brings no new information into the exchange.

- *Relevance maxim* (make your contribution relevant): Again, Mona's later contributions become more or less irrelevant as they are not informative and do not move the communication forward. Also Juli's turn (l. 36-37) is irrelevant to the course of the communication as a whole. However, she reacts to the teacher's question which springs out from the non-verbal reality of the teaching episode.

- *Maxim of manner* (be perspicuous and specifically avoid obscurity, ambiguity, be brief, be orderly): The teacher thinks that Mona on l. 5-6 is violating this maxim and therefore she asks her to write what she means on the board to explain the ambiguity/unclearness. There is no doubt that Suse is violating this maxim. Her turns are very long, she needs many words to express one idea, and there are repetitions and it takes her a long time before she gets to the point. (l. 17-30) What she basically says in her 14 lines is: "Peter's solution is right because it works with different numbers and Mona's number cannot be recorded and therefore doesn't exist." However, her turns always move the conversation forward.

(*Implicature*, i.e. what is inferred as additional meaning but not worded): Suse is in the position of an "arbiter"; she evaluates Peter's and Mona's ideas, says who means what and why this or that should be correct; in a way she seems to be stepping in for the teacher, as if the teacher could not understand.

Politeness principle:

- *Tact maxim* (minimize cost to others): A typical example in speech is the teacher's use of questions (l. 7, 34-35) and indirect questions (l. 12-13) rather than imperatives. These statements are obviously meant as commands.

- *Agreement maxim* (minimize disagreement, agree at least in part): Suse often obeys these principles at the cost of cooperative principles. One of her turns begins "This is what I wanted to say …" (l. 17) as if she agreed with Mona but ends "you cannot write it down" (l. 21) … "Thus a figure doesn't really exist." (l. 29) Also on l. 30 she says "…this could be right" although she basically means "this is right". The conditional is used here not to hurt Mona.

- *Sympathy maxim* (minimize antipathy between self and others) is manifested by Mona, e.g "I <u>only</u> meant" (l. 31).

ANALYSIS OF CLASSROOM DISCUSSIONS

Margarida César

The first thing that strikes us is that these students are already used to participate in this type of general discussions. This is illuminated by the way they react to their peers' interventions, trying to (re)interpret them, or complete and/or clarify what they stated, and also by the few times the teacher chose to make her interventions. This discussion shows part of the didactic contract of this class. This teacher is giving the students time and space to participate as legitimate participants (Lave & Wenger, 1991) and she is trying to develop a learning community. But this general discussion also illuminates the existence of an intersubjectivity that was developed between this teacher and her students (e.g., they all talk about the figures inbetween, and they know what they are referring to).

In this discussion there are two groups of argumentations: (1) the ones who argue that 0.9(9) = 1 is true (Peter, Suse, Jens); and (2) those who argue that this should not be true (Mona). But the point of this discussion was not merely finding a solution to this mathematical task. If that was what this teacher had in mind, students would not be used to this kind of general discussion. What this teacher wanted to do was to explore students argumentations and to facilitate students' appropriation of mathematical knowledge through discussion, i.e., through the diverse argumentations and confrontations that were elaborated by the students. This is, in our interpretation, why there are no evaluative comments on her talks. Even when she is trying to control Juli and Judith's behaviour (Lines 34 and 35), she does not produce an evaluative comment, and she does not use an imperative verbal form. She just tells them that everyone needs to be able to hear them, which is a particular way of interacting with students and making them pay attention and participate.

This general discussion illuminates different levels of cognitive development and also different levels of mathematical argumentation. Although most students use formal reasoning in their statements, Mona is probably at an interface between concrete and formal reasoning. This is probably why she believes there is another figure between 0.9(9) and 1, but also why she needs to go back to a more concrete description of that figure ("many many zeros", instead of "zero point infinite zero and then one"), but also why she needs to make the distinction between what can be said/thought (the figure she imagined) and what can be written down/drawn (Lines 8 to 11). And for her there are mathematical (logical) entities that can be imagined, that exist logically, but which can not be written down. According to her Talk 6 (Lines 14 to 16) she does not seem to have recognised any error in her previous statements. Probably the laughter (Line 10) is more a nervous sign than the recognition of a mistake. She seems to be trapped because she can imagine that figure – and the figure is very clear to her, mentally – but she is not able to write it down and she knows the rules of their game: if a figure cannot be written, then it does not exist. But for her, that figure could have a logical explanation, and according to her argumentation logic should be

accepted in Mathematics. This is why we interpret her laughter as confusion, disappointment, and not as the recognition of a mistake. Even after Suse's intervention in Talk 8 (Lines 23-30) Mona still thinks that figure exists logically, it just cannot be written down, and that is why it would not work. But she never claims that the figure she imagined would not be a periodic continued one.

Thus, Mona seems so taken by imagining the figure that would confirm her hypothesis that she forgot what is a periodic continued. She seems to be moving from concrete reasoning into formal one. As she is making an effort to imagine the figure between 0.9(9) and 1, she forgets the notion of periodic continued, that should be taken into consideration. But this way of reasoning – concentrating on one feature and not taking into consideration the others – is also very typical of concrete reasoning. It cannot be taken simply as a mathematical error, or lack of mathematical knowledge.

Suse is clearly using formal reasoning in her argumentations. She is able to make transitions between her own way of reasoning and Mona's argumentation; she is able to use Mona's language and then transforms it into more accurate mathematical language (Lines 23 to 30) and she is also able to use other examples to make her point clearer (Lines 24 to 26). She is also the one who explains to Mona that if we have a periodic continued, suddenly there is not a one in the middle of the zeros (Lines 39 to 41). Thus, she is the one who is able to argue in such a way that Mona will understand her point. And although Jens had also used a similar argumentation in his talks (Lines 1 to 4; and 38), he used his own argumentation and he did not relate directly to Mona's doubts/ difficulties. Thus, it was through Suse's interventions that Mona could be aware of some weaker arguments she used and replace them by more robust ones.

Just taking in consideration this small piece of interaction, I would say, if we wanted to use it for teacher evaluation, that her way of acting is very consistent and that she is able to develop students' participation, level of argumentation, respect towards each others' argumentations and autonomy. And these are competencies students need in order to succeed in evaluations (namely the most formal ones, like tests and exams) and also in their professional life. Moreover, she is able to facilitate students' mathematical development, as they do not merely repeat answers or rules they do not interpret, but they are developing their relational knowledge (Skemp, 1978).

THE PRODUCTION DESIGN OF "A FIGURE IN-BETWEEN"

Birgit Brandt

In Brandt (this volume) I outlined our concept of participation in mathematical classrooms (Krummheuer and Brandt 2001), which traces back to Goffman and Levinson. With respect to the interactional theory of learning mathematics, the main focus of our approach is the emerging process of 'taken as shared' meanings, which

includes the alternating of the active speakers and the interweaved emerging of the subject matter. Applying the production design to the transcript of Cohors-Fresenborg & Kaune, I will point out this interweaving for the interactive argumentation by the formulation "a figure in-between".

In the beginning, Jens refers to Peter, but he does not address him as a dialog partner – Peter is only one recipient of the broad listenership. Jens' contribution can be seen as a recapitulation and appreciation of Peters statements, but due to the presented extract it is not possible to decide about the production design of his utterance in detail. The argumentative *ideas* of Jens utterance are

- Between two digital numbers must be at least one figure.

- There is no figure between $0.\overline{9}\ldots$ and 1.

- Therefore $0.\overline{9}=1$ is logical.

In the ongoing interaction, these *ideas* are linked to Peter (e.g. [24]). So, Jens is surely not an *author* of all aspects of his utterance, but probably for the evaluation of this argument as logical. In contrast to Jens, who stresses his conformity to Peter, Mona emphasizes her autonomy. She explicitly refers to her responsibility (I do think), but she links her utterance to Jens's *formulation* that there always has to be a figure in-between [2]. She takes this part as a *ghostee* (that here is a figure [5]), and as an *author* she supplements a figure in-between $0.\overline{9}$ and 1, that doesn't exist [9, 16]. With her construction zero point infinite zero and then a one, some time or other [6] (and [16] as a *spokesman* of herself) she describes her certain *idea* of "a figure in-between", which she makes more explicit later as a *spokesman* of herself "I meant the figure that you would need in order to make zero point periodic continued nine a one" [14]. This *idea* of "a figure in-between" refers to the conception of real numbers as length of lines. Summarizing her several statements, these are the *ideas* of her argumentation:

- There must be a figure in-between in the sense of $0.\overline{9}+x=1$.

- The (not existing) figure zero point infinite zero and then a one (at the end of the unlimited figure) can be thought as this figure in-between $0.\overline{9}$ and 1.

At first, Suse is a *spokesman* of Monas *ideas* [17]. Subsequently, she continues with an additional example for a number in-between (three is in-between two and five in [24]). This can be seen as an application of Monas *idea* as mentioned above, but as a *ghostee* she uses this for an extension of Peters argumentation: There is no number in-between $0.\overline{9}$ and 1 in this sense, because Mona figure doesn't exist [30] (this is amplified in [36-44]).

First of all, Jens uses the *formulation* "a figure in-between" for his summery of Peters argument, but without clarifying his concept of in-between. Taking this *formulation* for a counter-argument, Mona explains more and more precise her *idea* of in-between. At the end, Suse ties up to Monas *idea* as a backing for Peters argument.

This interweaving is retraced by the reciprocal referring as *spokesmen and ghosthees*. Overall, the interaction process features the criteria of an "Interaktionale Verdichtung" (Krummheuer and Brandt 2001; "condensed period of interaction" Krummheuer 2007) – hence this interaction process provides optimized conditions for the possibility of mathematical learning.

REMARKS ON BOERO & CONSOGNO

Elmar Cohors-Fresenborg & Christa Kaune

Boero & Consogno (this volume) show how increasing mathematical knowledge can be constructed by social interactions. The mechanisms described by them are especially promoted in a discursive teaching culture. Activities like monitoring and reflection play a particular role. It is therefore obvious to analyse their transcripts also by means of the category system, which has been developed by Cohors-Fresenborg & Kaune (this volume) for the analysis of discursive and metacognitive activities. The connections of differing theoretical frameworks is meant to show exemplarily how scientific development in mathematics education can be promoted by international co-operation.

The categorisation of the two following transcript extracts are visually supported by colours, i.e. discursive activities are green, monitoring activities red and reflective activities ochre. Statements which do not match any of the categories remain black.

	Elisa:	I agree with Mattia, as he considers the results.	DS2a rMS4a
2	Giulia:	Mattia has considered all possibilities, because he has considered the two dice and has put the results and (I think) has looked at all possibilities.	DS2a rMS4b
4	Teacher:	Is it the same thing to think of the result or to think of the two dice?	DT1a
	Mattia:	It is the same thing ... no... yes!	MS8c
6 8	Giulia:	If you think of dice... to the digit shown by your dice... because the result is one digit plus another digit that makes a result. Before adding them, those two numbers are alone, they are not together... because if one casts 3 and the other 4.	DS2c MS8b
10 12	Roberto:	for instance, 4 is a number and 3 is another number, **as Giulia told**, if you add them, they make 7, but before putting them together, 4 is a solitary number and 3 is another solitary number, then when they go together we get a number formed by smaller numbers.	DS2d **DS2a**
	Giulia:	yes, but before getting the result, the two numbers can be other numbers.	rMS4c

The meaning of discursive activities for the social construction of knowledge can primarily be recognised in our analysis by a high share of green colour of the pupils statements, especially because it is solely sub-categories of DS2 that appears. This points to an "embedding of discursive contributions". The only intervention by the

teacher (line 4) is to be considered as an educational action, i.e. the invitation to a discourse (DT1a).

The red colour, which is the only other colour apart from green, shows that all other pupils' contributions are monitoring activities, i.e. careful supervision of their own (MS8) or other argumentations (MS4). The high share in founded metacognitive activities - marked by a prefixed *r* (*reasoning*) - is striking.

In the second transcript as well, most of the assigned categories belong to discursive activities. The sub-categories are spread similarly to the first transcript extract. The monitoring activities often contain reasons. A reflecting evaluation of a proceeding (RS6a) is new (lines 9-12).

Line	Speaker	Utterance	Code
2	Anna:	I agree with Giulia that in the 2-paths labyrinth you get out earlier, in the case of the 6-paths labyrinth you must try all the paths and you spend a lot of time.	DS2a
4	Matteo:	But in the 6-ways labyrinth you do not need to try all the paths, because for instance the first time you fail the exit, but then at the second or third trial you may find the good way to escape... You don't need to try all the paths!	*r*MS4c
6, 8	Giovanni:	It is necessary to consider the condition posed by Giulia, **namely that there is only one exit,** otherwise all the paths might have an exit, and it would not be a labyrinth any more!	DS2c **DS2d** *r*MS4c
10, 12	Mattia:	If a labyrinth would have more exits than paths with no exit, practically it would be very easy to escape, on the contrary if the labyrinth has the same number of paths and exits, ... it would be easier but the exits must be more than one half of the number of the paths.	RS6a
	voices:	less than one half!	MS2a
14	Teacher:	I would like Mattia to repeats his sentence - please, listen to him, then we will discuss what he said.	DT1a
16, 18	Mattia:	Can I make an example? In the 2-paths labyrinth there is one exit, while in the 6-paths labyrinth there are 3 exits; in order to make the 6-paths labyrinth easier than the 2-path labyrinth, you must put exits to more than one half paths, because if in the other labyrinth there are two paths and one exit, it is one half.	DS2c

REFERENCES

Goffman, E.: 1972, *Interaction ritual: Essays on face-to-face behaviour.* Harmondsworth, Middlesex: Penguin University Books.

Grice, H.P.: 1975, 'Logic and conversation', in Cole, P. and Morgan, J. (eds.) *Syntax and semantics*, vol. 3. New York: Academic Press.

Krummheuer, G. and Brandt, B.: 2001, *Paraphrase und Traduktion*, Beltz, Weinheim.

Krummheuer, G.: 2007, *'Argumentation and participation in the primary mathematics classroom.'* Journal of Mathematical Behaviour, (in print).

Lave, J., & Wenger, E.: 1991, *Situated learning: legitimate peripheral participation.* Cambridge: Cambridge University Press.

Leech, C.: 1983, *Principles of Pragmatics*. London: Longman.

Morgan, C., Ferrari, P., Johnsen Høines, M., Duval, R.: 2006, Language and Mathematics, Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education, Sant Feliu de Guíxols, Spain – 17 - 21 February 2005, 789-798. http://ermeweb.free.fr/CERME4/

Rowland, T.: 2000, *The pragmatics of mathematics education: Vagueness in mathematical discourse.* London: Falmer Press.

Skemp, R.: 1978, Relational understanding and instrumental understanding. *Arithmetic Teacher*, November, 9-15.

APPENDIX

Jens:	... but Peter says that zero point nine is the same as one and that otherwise there always	DS2d
2	has to be a figure in-between ... that there has to be at least one figure between two	MS8b
	decimal numbers. **And in this case, it isn't** and therefore it is logical that this should	**MS4a**
4	actually be correct.	MS4b
Mona:	Well, I do think that there is a figure. It may, however, be zero point infinite zero and	DS2f
6	then a one, some time or other	DS5d
	[she laughs] ...	MS8b
8 T.:	Could you please write it on the board, how you imagine this (figure)?	*d*RT5e
Mona:	No, not really, ... as a figure it doesn't exist ... you can't write it down in this way. But	MS4c
10	logically it would be possible.	*r*RS7
	[The following 38 sec. (Mona's slip of tongue) have been deleted.]	
12 T.:	I would like to know if it is at all clear to everybody what Mona wanted to say, about	*d*DT2c
	what figure she has been talking. **She said: "I cannot write it down."**	**DT2d**
14 Mona:	Well, I meant, the figure that you would need in order to make zero point periodic	DS2c
	continued nine a one. That's the figure I have been talking about. If first there are many,	
16	many zeroes and then at some time or other a one, but this doesn't exist in principle.	MS8c
Suse:	This is what I also wanted to say: There is an infinite number of nines behind the zero,	DS2d
18	hm, periodic continued nine, ... and she thinks that, that there should be a figure which	MS8b
	has exactly as many zeroes, which means infinite zeroes, ... well there is a one at the	DS2d
20	end so that, if you add it, you obtain one. That's the figure she is looking for. But you	*r*MS4c
	cannot write it down because there would have to be an infinite number of zeroes.	
22 T.:	Yes, let's pick up some more ideas. You said, we have got two positions. (...)	DT5c
	[No reaction of pupils during the next 14 seconds.]	
24 Suse:	I would say Peter's solution is correct, because, if you take different figures instead, if	*r*MS4b
	you take for example five and two instead of zero point periodic continued nine and	
26	one, you know that they are not the same, as three is between them. And regarding zero	
	point period continued nine and one, there is no figure between them. You know you	
28	cannot write down a figure. Mona, however, thinks that there should be this "zero point	DS2b
	period continued zero one"-figure, but you cannot write it down.	DS5d
30	Thus, it doesn't really exist a figure. And therefore this could be right.	*r*MS4c
	Mona.	DS1a
32 Mona:	Well, I only meant: the figure doesn't exist, but logically you could imagine it so	DS2d
	[laughter] That it could exist. Therefore ... *[murmuring]* .... **but this figure doesn't**	RS7
34	**exist. Hm, that is clear. It doesn't work.**	**RS6a**
T.:	Say it loud, please, Judith and Juli?	
36 Juli:	Yes, Judith and I are just trying to imagine the figure "zero point periodic continued	RS7
	zero one", but this is somehow weird.	DS2b
38 Jens:	I think that there cannot be a further figure behind a periodically continued figure.	MS2b
Suse:	Yes, that is right, yes, that is true, because the zero, **hm, because the periodic line is**	*r*MS2b
40	**above it,** which means, it is the zero that always repeats. Thus there cannot suddenly	**MS8c**
	be a one behind it.	DS2c
42	Which means this figure doesn't exist.	RS6a
	If at all, the periodic line would have to be above both figures, and then it would	**RS3a**
44	**be continued in that way: zero point zero one zero one zero one.** This would not be	RS6a
	the figure Mona meant.	DS2c

Jens:	... but Peter says that zero point nine is the same as one and that otherwise there always	DS2d
2	has to be a figure in-between ... that there has to be at least one figure between two	MS8b
	decimal numbers. **And in this case, it isn't** and therefore it is logical that this should	**MS4a**
4	actually be correct.	MS4b
Mona:	Well, I do think that there is a figure. It may, however, be zero point infinite zero and	DS2f
6	then a one, some time or other	DS5d
	[*she laughs*] ...	MS8b
8 T.:	Could you please write it on the board, how you imagine this (figure)?	*d*RT5e
Mona:	No, not really, ... as a figure it doesn't exist ... you can't write it down in this way. But	MS4c
10	logically it would be possible.	*r*RS7
	[*The following 38 sec. (Mona's slip of tongue) have been deleted.*]	
12 T.:	I would like to know if it is at all clear to everybody what Mona wanted to say, about	*d*DT2c
	what figure she has been talking. **She said: "I cannot write it down."**	**DT2d**
14 Mona:	Well, I meant, the figure that you would need in order to make zero point periodic	DS2c
	continued nine a one. That's the figure I have been talking about. If first there are many,	
16	many zeroes and then at some time or other a one, but this doesn't exist in principle.	MS8c
Suse:	This is what I also wanted to say: There is an infinite number of nines behind the zero,	DS2d
18	hm, periodic continued nine, ... and she thinks that, that there should be a figure which	MS8b
	has exactly as many zeroes, which means infinite zeroes, ... well there is a one at the	DS2d
20	end so that, if you add it, you obtain one. That's the figure she is looking for. But you	*r*MS4c
	cannot write it down because there would have to be an infinite number of zeroes.	
22 T.:	Yes, let's pick up some more ideas. You said, we have got two positions. (...)	DT5c
	[*No reaction of pupils during the next 14 seconds.*]	
24 Suse:	I would say Peter's solution is correct, because, if you take different figures instead, if	*r*MS4b
	you take for example five and two instead of zero point periodic continued nine and	
26	one, you know that they are not the same, as three is between them. And regarding zero	
	point period continued nine and one, there is no figure between them. You know you	
28	cannot write down a figure. Mona, however, thinks that there should be this "zero point	DS2b
	period continued zero one"-figure, but you cannot write it down.	DS5d
30	Thus, it doesn't really exist a figure. And therefore this could be right.	*r*MS4c
	Mona.	DS1a
32 Mona:	Well, I only meant: the figure doesn't exist, but logically you could imagine it so	DS2d
	[*laughter*] That it could exist. Therefore ... [*murmuring*] .... **but this figure doesn't**	RS7
34	**exist. Hm, that is clear. It doesn't work.**	**RS6a**
T.:	Say it loud, please, Judith and Juli?	
36 Juli:	Yes, Judith and I are just trying to imagine the figure "zero point periodic continued	RS7
	zero one", but this is somehow weird.	DS2b
38 Jens:	I think that there cannot be a further figure behind a periodically continued figure.	MS2b
Suse:	Yes, that is right, yes, that is true, because the zero, **hm, because the periodic line is**	*r*MS2b
40	**above it,** which means, it is the zero that always repeats. Thus there cannot suddenly	**MS8c**
	be a one behind it.	DS2c
42	Which means this figure doesn't exist.	RS6a
	If at all, the periodic line would have to be above both figures, and then it would	**RS3a**
44	**be continued in that way: zero point zero one zero one zero one.** This would not be	RS6a
	the figure Mona meant.	DS2c

DRIVING SPONTANEOUS PROCESSES IN MATHEMATICAL TASKS

Rossella Ascione and Maria Mellone

Dipartimento di Matematica e Applicazioni – Università Federico II di Napoli

(Italy)

In a linguistic perspective for mathematics learning, we illustrate how the Theory of Relevance, formulated by Sperber and Wilson in the ambit of cognitive linguistics and supported by new insights from neuroscience, can be useful both to interpret students' cognitive behaviours and to devise an effective didactical mediation. During the PDTR project we carried out an experimentation in an algebraic context in order to explore the impact of the planned didactical mediation on the spontaneous processes.

Introduction and theoretical framework

Sfard (2003) claims that

> "*the culturally tinged, but essentially universal, need for meaning, and the need to understand ourselves and the world around us, came to be widely recognized as the basic driving force behind all our intellectual activities.*" (p. 356).

The research of meaning and understanding within culture seems to be the very cause of the scientific development, the force driving behind intellectual human action. This is the starting point of Arcavi (2005), where the author through some examples infers:

> "*the developing the habit of sense making may be strongly related to the classroom culture that supports or suppresses it and is not merely an issue of -innate mathematical ability-*" (p. 45).

One of these examples, in particular, is about a mathematically able student and how, about a year after he finished the school, he solved a problem driven by the habits he had developed in his classroom. This habit seems to be very deep rooted; this is a habit of having a well-designed plan, including symbolic procedures to follow, without sense-making, which is invoked only if absolutely necessary. Therefore, the classroom culture has a central role in behaviour or habit of the learners and this has important implications for the didactical practice. Arcavi gives some advice in what direction the didactical practice could move, he suggests *"to develop the habit not to jump to symbols right away, but to make sense of the problem, to draw a graph or a picture, to encourage them to describe what they see and to reason about it."*(Arcavi, 2005, p. 45)

New insights from neuroscience research frame the problem of sense-making in a different prospective: in our brain the *absence of meaning* doesn't exist. According to Rizzolatti (2006), Changeux (2002) and many others[1] the way our brain works is subjected to an automatic and sometimes unconscious dynamic of search for meaning led by the target.

An explicatory example of this spontaneous need for sense-making is given by the famous problem of the captain's age, let us take the Dehaene's version (1997): "*On a boat there are 13 sheep and 12 rams. How old is the captain?*" (p. 153). This problem is really given in different primary classes where many children had diligently done the sum in order to answer: "the captain is 25 years old". Many studies have been carried out about this problem. We have carried it out, too. In some classes, to the request to motivate the answer, the pupils have given the following answers: "I have done the sum because the captain receives an animal for each birthday" or "I know the teacher wants the sum". These answers show that there is always a search for meaning and how this search is different among different persons.

In the linguistic and cognitive field, Sperber and Wilson use this brain's spontaneous aptitude for the search of meaning to build a more general cognitive and communicative theory. Sperber and Wilson (1998) think the search for a meaning is led by the relevance theory: "*...that is based on a fundamental assumption about human cognition: human cognition is relevance-oriented; we pay attention to the inputs that seems relevant to us*" (p. 8). The inputs are not just considered relevant or irrelevant; when relevant, they are more or less so, so relevance is a matter of degree, a relatively high degree of relevance is what makes some inputs worth processing, i.e. such inputs yield comparatively higher cognitive effects:

> "*The greater the cognitive effects, the greater the relevance will be. Cognitive effects, however, do not come free: it's necessary some mental efforts to derive them, and the greater the effort needed to derive them, the lower the relevance will be*" (Sperber & Wilson, 1998, p. 7).

In this way relevance is characterized in terms of cognitive effects and mental efforts. Summing up, "the Relevance-guided comprehension procedure", employed to a cognitive and a linguistic level, consists in:

> " *a. To follow a path of least effort in constructing and testing interpretive hypotheses (regarding disambiguation, reference resolutions, implicatures, etc.).*
>
> *b. To stop when your expectations of relevance are satisfied* " (Sperber & Van der Henst, 2004, p. 235)

The "comprehension procedure" provides for the existence of an effortless data processing driven by survival[2], we name them *spontaneous processes*. Also the new

---

[1] See for instance Gallese and Lakoff (Gallese & Lakoff, 2005).

[2] We mean by survival not only the one of the species, but also the survival of individual in a context.

insights from neuroscience show that there is a strategy that our brain has developed during centuries to allow those immediate decisions that have supported and support the survival of individual and species[3].

In the context of the problem of the captain's age we can see this kind of dynamic in the answer "I know the teacher wants the sum", while in the Arcavi's example mentioned above the dynamic is recognizable in the algorithmic way in which the student solved the problem.

These ways of reasoning are usually considered *meaningless*, on the contrary they are *spontaneous processes*. The teacher should always pay attention to these natural and spontaneous ways of thinking. He should make the student move from these *spontaneous processes* to a direction of higher cognitive effects even if this requires higher mental efforts: how could the teacher support this *shift*?

According to the Theory of Relevance these *spontaneous processes* take place if there is no alternative target recognisable as cognitively relevant. Above all, a way in which this shift becomes possible is to make the object of study relevant for the learners. The learners should feel the strong relevance of the cognitive target in order to make stronger mental efforts.

Classroom culture has a central role in these dynamics, as underlined by Arcavi (2005), but in the light of these theoretical outlines, it can't suppress the brain's spontaneous aptitude to the search of meaning but it can influence the way in which this search takes place. We hypothesize it is possible to create a classroom culture that supports the *shift*, a *meaning-oriented* classroom culture.

We believe there could be guidelines to describe this *meaning oriented* classroom culture. A good method could be the use of problems selected in order not to appear too trivial (so that the students can expect positive cognitive effects), but at the same time not too difficult, so that they will accept the challenge to solve them (this will give cognitive advantages to the efforts). During the solution of the problems it is important to give the learners the opportunity to be free to communicate their doubts, to express their thoughts and to give their own solutions.

The perception of the relevance of the input is different from student to student and for this reason in some situations the teacher should reinforce the students' self-esteem and encourage them to experiment and to explore a field trying to put it in order, even if not all is clear at the beginning. The teacher should always convey supposition of relevance to the task. For these reasons the learners should trust the teacher and should believe that he is proposing something interesting and within their reach, even if at the beginning it appears strange or difficult[4].

---

[3] See Damasio (1994).

[4] Mediator in accordance with L. S. Vygotskij

Methodology

We have carried out an experimentation, during the academic year 2005-2006, on two different sample groups of university students enrolled in the degree course of education, in order to explore the impact of the *meaning-oriented* classroom culture on the spontaneous processes.

The first sample is made up of 115 students who are going to attend the annual course of mathematical education, while the second sample is made up of 96 students who have just attended the same course.

Both samples share the fact that they were exposed to a traditional classroom culture[5]; furthermore, the second sample, during the above-mentioned course, had been exposed to a kind of *meaning-oriented* didactics.

We have engaged them in the following problematic situation:

Consider 4 consecutive numbers. Multiply the two middle numbers, then the two extreme ones and observe if any regularity occurs. If you find such a regularity, try to prove it by using a suitable language.

This particular task is supported by the following reasons:

- natural numbers are a privileged context for exploration because of their content of innate knowledge;

- the use of algebraic language is not univocal, but it is determined by the individual target;

- using algebraic language, the learners can activate *spontaneous process*;

- the need for a proof should represent the target to support and guide the effort to achieve the *shift*;

- this kind of problem is not trivial for our students, but at the some time is not so difficult.

Before facing the problem, students are encouraged to write down on a valuable "*data collection*" instrument: the *board diary* of this math experience. In order to fulfill this task, students are asked to note down the thinking process involved in making hypothesis, trying to answer the question proposed. For this reason, the board diary becomes a valuable instrument to collect data and to explore in detail the different reasoning approaches of the students. Furthermore the board diary works together with the *sense-oriented* classroom culture.

Despite the incidence of *spontaneous processes*, we predicted that the influence of *meaning-oriented* classroom culture on the second sample would substantially reduce the *spontaneous processes*.

---

[5] It is the one in which the teacher just passes instructions to the students about contents.

The collected data made us go on with our research; we interviewed two girls belonging to the second sample group in order to understand the dynamics that lead to a conscious control of *spontaneous processes*.

Experimental data

From the analysis of the board diary, we have drawn out the following approach to the problem: in both the sample groups, in a first phase, all the students tried to understand what kind of regularity could be hidden by four consecutive numbers. After discovering it, many of them tested the regularity revealed with other sets of four consecutive numbers, and finally they expressed it in natural language. Someone tried to verify the regularity by using rational numbers, focusing the attention on the meaning of consecutive numbers.

The problem of communicating the regularity with a suitable language arose in a second phase. The students looked for a "formal language", i.e. for suitable symbols by which to express the rule.

In the following table we have singled out three different types of solution to the problem. The analyzed types are listed together with the number of the answers and the corresponding percentage of those who have employed them to solve the task.

Table 1

Types of solutions	1	2	3
	If a,b,c,d are consecutive letters/numbers, it happens that $m =$ product of the extremes, $n =$ product of the middles $(a{\cdot}d) = m$ $(b{\cdot}c) = n$ $n>m$ *with/without* $n-m=2$	Utilizing four letters a, b, c, d to represent the numbers, they put $(b-a)=1$ $(c-b)=1$ $(d-c)=1$ and after they wrote the rule $b\,c > a\,b$ $(b \cdot c) - (a \cdot b) = 2$	Starting from a, $(a+1)$, $(a+2)$, $(a+3)$ the expression of general consecutive numbers, the regularity can be expressed by $(a+1){\cdot}(a+2) > a{\cdot}(a+3)$ $(a+1){\cdot}(a+2) - a{\cdot}(a+3) = 2$

Percentage of answers for the first sample	80 answers	30 answers	5 answers
	69,6%	26%	4,4%
Percentage of answers for the second sample	49 answers	20 answers	27 answers
	51%	20,9%	28,1%

In the first approach, we can detect a *spontaneous* way of processing data. It's a kind of writing that merely "translates" information from natural language. This last aspect is underlined by the absence of translation "into letters" the property that the numbers are consecutive.

These behaviours are driven by the *spontaneous processes*. Moreover, in many board diaries that present this type, the students believe to prove the regularity by simply verifying it by using a congruous number of examples.

In the second sample group the percentage of these answers is smaller than the corresponding percentage in the first group. However, it was not as huge as we expected. In fact, although we were convinced that some *spontaneous processes* would prevail, we did not expect that they would affect in such a way also those students who had been subjected for one year to *meaning-oriented* classroom culture.

The second type is similar to the first one even if there is an attempt to write the property of consecutive numbers using the algebraic language. Once more, the link with natural language is very strong. The expression b•c > a•b shows how students express the regularity with the natural language, that is to say "the product of the mean terms is bigger than the product of the extremes one by two", they feel the communicative target relevant, because the target of proving is too far to grasp.

Perceiving the relevance of the input of proving, other students try to prove the property, but they assert that the formulation they carried over was "too hard to be solved". They note down in their board diaries, "I have tried to find an unknown quantity to substitute in the general expression… but I don't succeed, there are always a lot of variables".

Also in this case, the percentage of answers of the second sample group is a little bit smaller than the corresponding percentage of the first group. This type of solution may be seen as a bridge between the other two types of solutions; in fact, it contains both those who have the target of proving and those who have a communicative one.

The third type of solution shows that the students who employ it have grasped the aim of the proof. The aim of the proof is evident in both sample groups where there is a certain number of students (4,1 % in the first one, 19,1% in the second one) who have rejected solutions of the first or the second type, in order to chose the third type

of solution in the name of the relevance task of proving. Moreover, it's worth noticing that only 2.6% of the first group and 15.6% of the second one achieve a formal proof. Once the students found the right language to prove, they forget the necessity to prove the regularity.

Although we did not expect a huge increase in percentages between the two sample groups, once more it is worth noticing that between the first and the second group the difference for the third prototype is not very large.

After a year of course characterized by *meaning-oriented* classroom culture, many students still retain some *spontaneous processes* and they decided to invest few efforts in solving the problem (71.5% of the answers of the first and of the second kind).

In order to understand better the reasons that allow the *shift*, we interviewed two girls belonging to the second sample group. M employed the third prototype of solutions, while N employed the first one. M has been chosen because she did not carry out the task of proving, even if she showed the will to prove the regularity in her board diary. N, instead, has been chosen because of the complete absence of an attempt to prove in her diary.

The interviewer asks some questions in order to comment on their solutions.

Table 2 (interview with M)

06	I: Why did you stop?
07	M.P.: I stopped because… it could be expanded, anyway. Well, usually when I deal with such things I get discouraged.
08	I: Don't you want to try?
09	M: Ok… [makes calculations]. Ah, it's an identity.
10	I.: Why did you choose this kind of algebraic formulation?
11	M: At the beginning I was perplexed to write a+2 and a+3 because it seemed to me a little bit stupid, something very simple…for little children, but however it had sense so I used it, I have also thought to an alternative way to write the rule, maybe using a, b, c, d to indicate the consecutive numbers, but it seemed very laborious. I thought I could never use them because a, b, c, d didn't have any relationship at all. But I have adopted a, a+1, a+2, a+3 because it has only one unknown quantity, in some way there is all in that formulation… it seemed very useful since I wanted to use it to prove the regularity. I wanted to go on. Apart from translating the property of consecutive numbers, I had to demonstrate the regularity, so I have looked for a suitable language for my task.

Although M has chosen a solution of the third type, she forgets, like others students, the final task of proving the regularity and the interviewer has to help her in achieving the proof. Even if the words of the interviewer in line 8 are quite trivial, he plays a fundamental role: he gives confidence to the student supporting her in the effort of making calculations.

M is characterized by a strong metacognition that allows her to self-regulate her own thoughts. For instance, in line 11, she is perplexed by the use of a formulation so simple, but she understands that it is the best way to reach her declared aim of proving the regularity, so she uses it. She also declares that her first thought was to use the solution like the prototype 1-2, however she excluded it because it seemed too laborious to her: being equal the cognitive effects of the task, she chooses to follow a path of least effort.

Table 3 (interview with N)

07	I: Why did you use a, b, c, d as representation?
08	N: I though of letters written in a sequence.
09	I: How did you use this language to prove the regularity you found?
10	N: No, actually I didn't use it to prove because in this way I cannot multiply them, I wouldn't manage to see the rule, how do I know that b•c is bigger than a•d by 2? To know that I should transform them in numbers.
11	I: But in this way you go back to the starting point, you go back and verify it for each number. If we want to prove it we eventually have to use algebraic language. Can't you find another way?
12	N: [thinks for a few minutes] So I could do a, a+1, while c should be a +1+1, and the other should be a+1+1+1. Ok, now I found the letters so the product of "a times a+1+1" is bigger by 2 than " a+1 times a+1+1", [in the meantime she writes a•(a+1+1+1)>2, and stops] but, how do I write "than"?
13	I: What do you mean when you say that a number is bigger than another number by 2?
14	N: It means that the number equals to the other one plus 2. [after a while she writes a•(a+1+1+1)=(a+1)•(a+1+1)+2]
15	I: This is the rule you were talking about, are you sure it's true? How can we be sure it works?
16	N: I have the letters, I want to demonstrate it's true, so I make the calculations and verify that the quantities are the same. [expands and verify it is an identity]

17	I: Is it so difficult to change your point of view?
18	N: Yes, for me it's spontaneous to answer in that way, because when I deal with a mathematical problem I feel the urge to answer quickly, in spite, during the course the teacher told us more than one time to reflect and reason upon it.

In the above interview things are clearly more complicated. The interviewer led the interview in a more articulate way, assuming a stronger role of didactical mediator. He led N to think (lines 07, 08, 09, 10) about the inadequacy of the algebraic language she used and he invites her to look for a more useful language (line 11). N promptly reacts to the solicitations of the interviewer, just as if she has all the right answers locked in a drawer whose key is the question proposed by the interviewer. Sometimes the *spontaneous processes* get stronger again, as when N has problem in translating into algebraic language "bigger by 2 than"(line 12). In such moments the interviewer has to guide her with his speech (lines 13, 14).

Anyway N cannot clearly see the final aim, and it is the interviewer, giving relevance to the task proposed, who direct her towards the proof.

In line 17 the interviewer asks a strategic question stimulating a metacognitive process in N; as a result she claims, in line 18, to be victim of strong *spontaneous processes*.

Conclusions

The *spontaneous processes* of some mathematical behaviours, in particular algebraic one, are activated by survival in absence of other relevant input. Under this interpretation they sometimes represent a source of strength, but we think it is necessary to learn how to control and regulate them.

Our data clearly show that even in a *meaning-oriented* classroom culture situation the *spontaneous processes* are still present, and surprisingly, in a high percentage. Probably this last factor could mean that a single year of a similar type of didactics is not enough to regulate *spontaneous processes*. It is also important to notice that one year of *meaning-oriented* classroom culture supported N, as seen in line 18, to become aware that other ways of reasoning - different from the spontaneous one - exist. This awareness represents the first step towards the metacognitive capability that allows the *shift*.

On the contrary we observed in M's interview that the metacognitive capability allows her to see as relevant a target that had not been recognized as such at the beginning, and not worth to invest an apparently useless effort. In this way the metacognitive capability helps to regulate *spontaneous processes*.

Moreover we saw that in N's interview, where the metacognitive capability was weak, the expert supported the relevance of the target giving her the confidence that

her further actions, even if not totally clear at the beginning, would give her positive cognitive effects.

In this way the expert trains N to intellectual patience and helps her to acquire the necessary self-regulation to develop an autonomous metacognitive capability. This should be the teacher's precise role.

REFERENCES

Arcavi, A.: 2005, 'Developing and using symbol sense in mathematics', *For the learning mathematics* **25**, 2, pp.42-47.

Changeux, J.-P.: 2002, *L'Homme de Verité*. Éditions Odile Jacob, Paris

Damasio, A. R.: 1994. *Descartes' Error. Emotions, Reason and the Human Brain.* Putnam, New York.

Dehaene, S.: 1997, *La Bosse des Maths*. Éditions Odile Jacob, Paris.

Gallese, V. and Lakoff, G.: 2005, The brain's concepts: the role of the sensory-motor system in conceptual knowledge. *Cognitive Neuropsychology*, 22 (3/4), 455-479.

Iannece, D. and Mellone, M. and Tortora, R.: 2006, 'New insights into learning process from some neuroscience issues', *Proceedings of PME 30*, Vol. **3,** pp321-328.

Freudenthal, H.: 1983, *Didactical Phenomenology of Mathematical Structures*, Mathematics Education Library, D. Reidel Publishing Company, Dordrecht/ Boston/ Lancaster.

Rizzolatti, G. and Sinigaglia C.: 2006, *So quel che fai. Il cervello che agisce e i neuroni specchio*, Collana Scienze e idee, Cortina Raffaello editore.

Sfard, A.: 2000, 'Steering (Dis)Course between metaphors and Rigor: Using Focal Analysis to Investigate an emergent of Mathematical Objects', *Journal for Research in Mathematics Education*, Vol. **31**, No.3, pp.296-327.

Sfard, A.: 2003, 'Balancing the unbalanceable: the NCTM Standards in the light of theories of learning mathematics', in Kilpatrick, J., Martin, G. and Schifter, D. (eds), *A research companion for NCTM standards*, Reston, VA, National Council for Teachers Mathematics, pp.353-392.

Sperber, D. and Van der Henst J. B.: 2004, 'Testing the cognitive and communicative principles of relevance', Noveck, I. & Sperber, D. (eds) *Experimental pragmatics,* Palgrave, pp. 229-280.

Sperber, D.,Wilson, D.: 1998, 'Pragmatics and time', in R. Carston & S. Uchida (eds) *Relevance Theory: Applications and Implications*. John Benjamins: 1-22.

COMMUNITIES OF PRACTICE IN ONLINE MATHEMATICS DISCUSSION BOARDS: UNPICKING THREADS.

Jenni Back, Middlesex University and Nick Pratt, University of Plymouth

This paper uses the perspective of communities of practice to examine some data taken from the NRICH website discussion boards (www.nrich.maths.org.uk). It suggests that examining the interaction on the discussion boards in terms of different interpretations of communities of practice sheds some light on the nature of the mathematical learning that may be possible. Different sections of the same exchange are characterised by different interaction patterns and one seems to be closer to patterns of interaction that are typical of classrooms than the other. The contrasting section seems to reflect some aspects of collaborative problem solving. We suggest that it might be possible to encourage participants to make more contributions that are tentative and collaborative with support from the website.

INTRODUCTION

The increase in the use of the internet has been reflected in the burgeoning number of resources aimed at supporting learning, including online discussion. One such resource is the NRICH website (www.nrich.maths.org), set up in 1997 with the aim of offering school students the opportunity to engage in challenging mathematical activities. As well as accessing problems designed to 'enrich' the students' mathematical diet, students are able to post questions for others to comment on in discussion 'threads'. The threads that develop are also monitored by *moderators*, 'expert' (usually undergraduate) mathematicians who support the thread and are also responsible for vetting postings.

Whilst both authors of this paper would support the use of new technologies in educational settings, and indeed are enthusiastic about doing so, we share Latchem's concern (2005) that more needs to discovered about *how* they are being used. In this paper we take a socio-cultural approach based on Wenger (1998) in order to address our research question, namely: *what kinds of practices are afforded by online mathematical discussion boards?*

Our collaboration arose as the result of meeting at a conference where Jenni submitted a paper based on Wenger's (ibid.) theoretical framework of communities of practice that started to explore what she saw at that time as the community of practice of users of the NRICH website discussion boards. Nick felt that the perspective of *a* community of practice was too limited and did not tell the story as it was; in his view the participants were engaged in more than one community of practice. To explore this, we decided to take some sections of data from the discussions and code them independently to see what our differing perspectives brought to the analysis. Our initial analyses revealed similarities and differences: Nick seeing the ascendance of teacher-like strategies on the part of those acting as mentors and Jenni inclined to

analyse the use of mathematical and social skills and to view the setting as a mathematically collaborative one. We could each accept the validity of the other's view but our lenses were colouring our interpretation so we decided to try to unpick the threads of our analyses and bring them together into a coherent story.

THEORETICAL PERSPECTIVE

Whilst the NRICH website lies outside formal education in the sense that it is an open resource for both teachers and students, one of its stated aims is 'to foster a community where students can be involved and supported in their own learning and where effort and achievement is celebrated' (http://nrich.maths.org/public/viewer.php?obj_id=2712). Our interest, therefore, lies in how this 'community' is constituted; how students' involvement works and how it supports their learning. To explore these questions we have adopted a theoretical perspective based on Wenger's (1998 and also Lave & Wenger, 1991) notion of communities of practice (CoP). There is currently a good deal of interest in how this perspective can be used in educational research and in trying to understand learning situations in terms of the communities they represent. Yet in discussing the notion of community, Wenger is unclear as to whether the term is meant to represent an essential entity, (a thing that exists) or a way of understanding a situation (a construction that allows a new insight in some way). In writing that 'we belong to several communities of practice at any one time' and that 'communities of practice are everywhere' (Wenger, 1998, p. 6), the implication is that a community resides somewhere as an entity. On the other hand one of Wenger's central points is the 'double-edged' nature of *reification* which both affords and constrains practice, giving for example, 'differences and similarities a concreteness they do not actually possess' (ibid., p. 61). Applying this idea to the notion of community itself, one can become tangled up in questions about the existence, or otherwise, of particular communities, about whether individuals are 'in' them or not and what their boundaries are. For example, in relation to the NRICH discussion threads one might ask whether or not these are communities of mathematicians, communities of learners and teachers/mentors, or communities of social agents; and who belongs to which community.

To avoid this difficulty, we take a different approach by asking what viewing the discussion thread *in terms of* a community can tell us about it. This means we are not interested in whether the discussion thread is, or is not, legitimately identifiable as a community of practice but in how using the 'lens' of communities of practice offers fresh insights into the situation. We therefore see the CoP as being imposed by us on the situation; not constituted in any real sense *by* the situation.

Rather than just one lens though, we have tried to understand the discussion threads through the use of two lenses in the form of different, idealised communities of practice which act only as a vehicle for making comparisons (Pratt & Kelly, 2007).

The idealised communities we have chosen to construct are those of 'school mathematics' and 'research mathematics'. Such a distinction between two different forms of mathematical knowledge is founded in an understanding that people come to know things through the working practices of the context in which they learn (Boaler, 1997; 2002). Thus, school pupils come to know mathematics through the practices of schooling which tend to be focused on the 'educational discourse' of classroom interaction rather than the 'educated discourse' of the subject (Mercer, 1995). This educational discourse of school mathematics is characterised by social norms which focus participants on teacher authority, on *learning* rather than *doing* mathematics and on goals that are to do with those particular aspects of the subject which are rewarded by the assessment regime (for example Doyle, 1986; Mercer, 1995; Pollard, Triggs, Broadfoot, McNess, & Osborn, 2000; Pratt, 2006; Schoenfeld, 1996).

In schooling, pupils therefore develop very different forms of mathematical knowledge than, say, a professional research mathematician working collaboratively with colleagues. Whilst such a 'research' community might take many forms, we characterise it here in an idealised form as focusing on mathematical activity in which participants are all seeking 'a particular kind of knowledge [in which] "answers" are not generally known in advance', (Schoenfeld, 1996, p. 16). In this idealised community, 'the real authority is not the Professor – it's a communally accepted standard for the quality of explanations, and [a shared] sense of what's right' (ibid.). Lampert (1990, p. 33-34) has characterised this form of activity as follows:

> [participants] are courageous and modest in making and evaluating their own assertions and those of others, and in arguing about what is mathematically true; they move around in their thinking from observations to generalizations and back to observations to refute their own ideas and those of their classmates … they put themselves in the position of authors of ideas and arguments; in their talk about mathematics, reasoning and mathematical argument – not the teacher or the textbook – are the primary source of an idea's legitimacy.

Note that we do not hold up either of these models as 'ideal' in the sense of best. Rather, they are 'idealised'; abstractions which best represent some particular aspect of practice. In the case of research mathematics, we wish to focus attention especially on the joint enterprise of solving problems unfamiliar to the participants, where the answer is achieved together through joint negotiation. We contrast this with the 'school' situation where pupils are often required to find particular answers to questions posed, and already understood by the teacher. Other features of each idealised community are shown in figure 1.

By means of comparison with the chosen ideals we are able to comment on participants' practices in terms of the theoretical dimensions of 'community' that Wenger identifies. In particular, we use:

- the way in which participants identify themselves *with* certain practices, and *as* particular kinds of people;

- the notion of *belonging* and its three strands of *engagement* with and *alignment* to practices and *imagination* of possibilities;
- the criteria Wenger proposes for communities of practice: *mutual engagement* in a *joint enterprise*, making use of a *shared repertoire* of practices and experience.

Communities	Mathematics Classroom (Pupil)	Mathematics Classroom (Teacher)	Community of Research Mathematicians
Ways of knowing mathematics	- As a pupil.	- As a teacher.	- As a researcher.
Implicit and explicit goals of participants	- Individual pupil learning. - Recalling knowledge and performing on school tasks, achieving grades, gaining praise from teacher. - Identification as **expert pupil**.	- Successful learning by pupils. - Successful completion of tasks and achievement of grades in assessments. - Identification as **expert teacher.**	- Doing mathematics. - Sharing knowledge publicly through conferences. - Creating new knowledge together, gaining publication, gaining esteem of peers etc. - Identification as **expert mathematician**.
Model of expertise	- Successful in **learning** maths. - Successful participant in classroom practices. - Successful in assessment regime; able to work quickly; able to recall methods effectively.	- Successful in **teaching** maths - Successful at supporting children to complete tasks and facilitating learning. - Able to model practices associated with expertise in teaching.	- Successful in **doing** maths. - Successful in working with leading researchers in department and visiting researchers in substantive theory building; publishing work.
Forms of discourse	- Dominated by teacher control with requirement to respond to teachers' direct questioning. - Mainly 'educational discourse' (Mercer, 1995).	- Dominated by need to manage classroom environment. - Mix of 'child' and 'adult' discourses. - Mainly 'educational discourse'.	- Dominated by exploratory talk. - Other forms through a range of media (email, e-community, journals etc.). - Mainly 'educated discourse' (Mercer, 1995).

Figure 1 – Features of two idealised mathematical communities

DATA AND ANALYSIS

The data are drawn from two discussion threads that addressed problems of finding the nth term in a given series of numbers. This scenario is a common one in school mathematics from the end of Key Stage 2 onwards. As is usually the case, the thread is initiated by a student who poses a problem that is puzzling her:

> Mary: can somebody tell me the nth term of 8,11,17,26,38 thankyou

The person who is posting is a novice poster: this is the first time she has put a message up on the discussion boards. She asks politely, which is typical of the messages that occur here, and she asks the question very late at night: 11.53pm,

despite which someone answers her within a quarter of an hour. The exchange takes place in August, meaning that the problem that Mary is considering is fairly unlikely to have been instigated by a teacher in a school setting. The polite request is certainly part of the shared repertoire of the community of practice.

The response to Mary comes from Peter (see below). He offers some hints followed by some more after a break of 12 minutes. 'Hint offering' by someone adopting the teacher/mentor role can be seen as part of the shared repertoire. Mary fails to reply and the next day Peter asks why but again gets no response. All this raises some questions about the way in which the participants here drop in and out of the communities of practice in which they are engaging. The setting is characterised by fluidity and a degree of potential for participation and non-participation that would not be viable in face to face contexts. However the responses of Peter to Mary's question are typical of those found here:

Peter: Do you notice that the second term is 3 more than the first? And that the third is 6 more than the second? And each successive increase follows this pattern: 3, 6, 9, 12. So you would expect that the 6th term would be 15 larger than 5th, right?

5   Have you ever noticed that the sequence of squares follows a similar pattern? Look at this sequence:
1, 4, 9, 16, 25
This sequence has successive differences of 3, 5, 7, 9, ...
So your sequence is very similar to a sequence of squares, except that it

10   seems to be "expanded" by a factor of 1.5, so that instead of increasing by two's, the successive differences of your sequence seem to be increasing by three's.
Is this a good hint to help you get started? *(12 minute break)* If that doesn't help, consider this: subtract 8 from each element of your sequence. This

15   gives you

0, 3, 9, 18, 30
Now, divide each element of the sequence by 3, giving you
0, 1, 3, 6, 10
Does this look familiar? These are the "triangle numbers", which are the

20   number of, say, bowling pins you can arrange in triangles of different sizes:
0,

X,

X
XX,

25   X
XX
XXX,

X

30
XX

XXX

XXXX,

etc.

The formula for the number of bowling pins in a triangle with n pins along one edge is well known.

35
If you know it, then you almost have your answer.

In this part of the exchange, Peter's posts can be read as setting himself up as an expert teacher. He assumes the role of the teacher adopting patterns of behaviour that are well established in teaching such as asking rhetorical questions (line 1, 2, 4, 5 etc.), outlining routines and procedures (lines 5-8, 4-19), offering hints (lines 9-12), comparing to analogous situations (lines 6-8), allowing time for pupil responses and following these up with further more explicit hints (line 13). The development of increasingly detailed hints so that the learner's solution is supported more and more is a typical teacher strategy (Mercer, 1987; Back, 2004). Mary adopts the stance of a mathematical learner in asking her question so this initial exchange is best represented by a community of practice of teachers and learners of mathematics. In this community of practice the joint enterprise involves solving mathematical problems through the teacher/mentor using strategies, such as those described above, to support the pupil in doing so.

There is no pressure on anyone in this context for mutual engagement with the possible exception of the University student mentors and members of the NRICH team who are expected to monitor the discussion boards regularly. This contrasts with a school setting in which pupils and teachers are obliged to engage with each other to a greater or lesser extent.

Later on in the section of transcript that we considered there was much more evidence of collaborative problem solving in which all the participants seem to be working within a community of practice focused on mathematics: the community of research mathematicians described above. In this later part of the sequence participants offer suggestions as well as ask questions. Some of the questions that are asked are left in the air for long periods of time before they are picked up again and some are just dropped without being followed up. The following excerpt illustrates this:

Andrew: ....

for the sequence of n^2,

1,4,9,16,25,36,49,64,81... the second level of sucessive difference is 2.

But how can explain (sic.) why does this correspond to the formula n^2??

5   Jeremy: They correspond to different powers e.g, the first level is linear, the second quadratic, third cubic etc.

Although for n^3 the third successive difference is 6, and I can't think how it

corresponds (except to that the second level is 2, adn 2x3 (i.e. because it is cubed) =6) Yes, fourth level has a common difference of 24, which is 6x4,
10   so I think that's right.

Andrew: Jeremy, I think you misunderstood me...
               for eg,
               1,4,9,16,25,36,49,64,81...
               the first level is linear, that is,
15   3,5,7,9,11,13,15,17...
               2n+1
               the second level is somehow like this:
               $(2(n-1)+1)+(2(n-2)+1)+(2(n-3)+1)+(2(n-4)+1)+...+(2(n-n)+1)$,
               which = n^2
20   However, why??? How do you derive it from that??

Jeremy: Woops! I'll have a think…
               I'd guess that we'd need to take it further to derive an answer How about the third level in terms of n?

In this sequence there is an interesting contrast with the earlier transcript in that both participants make suggestions and also ask questions. Andrew's intervention is his first into this exchange but from his comments he seems to have been 'lurking' and watching what was going on. The sequence is fairly quick, being completed in less than an hour. Significantly, it seems to be an example of two people, neither of whom knows the answer to a question, puzzling it out together. The use of 'we' in Jeremy' final sentence seems to indicate a coming together as 'researchers' of this mathematical problem. The use of language here is much more tentative (lines 7, 20) with 'hedges' being used by both participants (lines 10, 20) and neither participant taking the lead. Each participant voices his own ideas using 'I' but also moves to using 'you' in a move to collaborate with the other. We would like to suggest that the discussion being undertaken here reflects far more a community of practice of research mathematicians than that of a school mathematics classroom.

CONCLUSIONS

Our research question explores the kinds of mathematical practices that are offered by on-line environments. Our findings suggest that the interaction on the discussion boards can be seen in terms of a number of different communities of practice and that the participants are able to shift between adopting roles of expertise in relation to mathematics as well as to its teaching and learning that are different to those in mathematics classrooms. Rather than declaring the discussion threads to be *a* community of practice and then trying to define its parameters, we have found it enlightening to compare the online environment to two idealised CoPs. Making use of Wenger's (1998) dimensions of community, belonging and identity, this comparison sheds light on the situation in a different way. Participants are *mutually*

engaged in a set of well defined practices, many of which appear to reflect strongly those of the classroom. Some posters identify themselves as novice mathematicians, putting up requests for help from 'experts'. Respondents may identify themselves as expert mathematicians, but the language they identify themselves with is usually that of expert teachers ("Do you notice that…?", "Is this a good hint to get you started?"). In this sense, the practice in which they are mutually engaged is largely teaching and learning, not collaborative problem solving. Their *joint enterprise* is the support of a novice in developing mathematical knowledge; learning mathematics, not doing mathematics. If the ways of coming-to-know this mathematics through the shared repertoire of the participants are replicating those of a classroom the novice may develop his or her expertise in a similar way to being at school. This may of course be a good thing if the goals of the community are to get better at school mathematics, but if the aim of an online community is to develop different ways of coming-to-know mathematics then the practices may not be well aligned with the goals – an issue that is at least significant.

On the other hand, having the idealised situation of 'research mathematics' in mind, outlined at the beginning of this paper, helps us to see occasions where the joint enterprise becomes an attempt for two people to understand something together (Andrew's and Jeremy' dialogue). Both participants here seem to identify themselves as co-investigators, exploring the idea in turn and making conjectures, trying things out and asking questions; all parts of the mathematical discourse that schools often fail to embed meaningfully in their work (Lampert, 1990). Their *mode of belonging* is different to that of other participants at this point. The absence of posts that 'teach' them the answer affords greater use of their own imagination and their alignment to hints and tips is temporarily suspended, affording them space to think in a different, perhaps more creative, way.

We do not want to suggest that one form of practice is better than the other per se. Having a space where students can get responses to mathematical problems with which they are struggling is a useful and important resource. We would note too that the analysis above is painting a picture that is too black and white. The online environment, as it stands, is still likely to be offering more opportunities for shifts in power relations between the participants and more opportunities to voice mathematical meanings and have these taken seriously than might be found in 'ordinary' classrooms. We should also note the over simplicity of a simple dualism between school and research mathematics. Not only might there be other 'kinds' of mathematics (different ways of coming to know it), but there are other communities of practice which could be used as lenses. Not the least of these is a community of social agents whose mutual engagement is in the process of online discussion itself. Interestingly, the NRICH discussion site encourages this by labelling participants in different ways depending on the extent to which they engage with the site, so that there are 'new', 'prolific', 'experienced' and 'veteran' posters. The practice of being online will itself be a significant part of the identity and belonging of participants.

Nevertheless, what we do want to point to is that the practices of the discussion threads are afforded and constrained largely by the environment itself. In saying this we do not want to imply any simple causality. Though there is some evidence that the nature of online discussion tends to militate against participants getting as far as 'building shared goals and purposes and producing shared artefacts', remaining instead at the level of 'articulating individual perspectives' (Murphy, 2004), individuals' own agency still allows participants to engage in the environment in many different ways. In practice, examples of non-aligned use can be found in discussion threads. However, these are not all that common, and threads generally operate around a well established historical set of practices involving a pattern of: post a query; offer hints and tips; refine understanding; offer further hints converging on a 'solution'; social rounding up. Such tight alignment to the 'rules' of engagement in the online environment tend not to afford practices that reflect those of the 'research' environment, affording instead those of the 'school'. They thus tend to cut out many of the mathematically important ways of thinking that schools find so difficult to develop in their students.

We would assert that the use of our two idealised communities supports some interesting observations about how the NRICH site affects the way students engage with, and hence come to know, their mathematics. In relation to our research question outlined at the start of this paper there is considerable evidence regarding how the NRICH website offers opportunities for informal, shared learning experiences and for the kind of collaborative, informal learning from each other that has been the centre of previous research questions relating to online environments (e.g Beetham, 2005). Indeed, the site can perhaps be seen as a model of good practice in this respect. However, by taking a socio-cultural perspective using communities of practice, this paper has suggested that such informal collaboration can entail a number of different forms of practice, each of which will lead to mathematical engagement that is of a different nature. What might be of further interest is to look at ways in which these practices can be deliberately influenced to shift the way in which participants make this engagement. This might be through promoting different guidelines for postings, perhaps encouraging people to hold back more from postings to which they already know the answer, or by moderators modelling different kinds of responses. In this way online discussions such as these might provide even greater alternatives to schooling than they do currently, not necessarily because such environments are 'better', but at least so that students have an alternative way to come to know mathematics.

Acknowledgement:

Bibliography

Back, J. (2004). *Mathematical Talk in Primary Classrooms: Forms of life and language games.* Unpublished Ph D, King's College London, London.

Beetham, H. (2005). e-Learning research: emerging issues? *ALT-J, Research in Learning Technology, 13*(1), 81 - 89.

Doyle, W. (1986). Classroom organisation and management. In M. Wittrock (Ed.), *Handbook of Research on Reaching*. New York: Macmillan.

Edwards, D. and N. Mercer (1987). Common Knowledge: The Development of Understanding in the Classroom. London, Methuen.

Lampert, M. (1990). When the Problem Is Not the Question and the Solution Is Not the Answer: Mathematical Knowing and Teaching. *American Educational Research Journal, 27*(1), 29-63.

Latchem, C. (2005). Failure - the key to understanding success. *British Journal of Educational Technology, 36*(4), 665-667.

Lave, J., & Wenger, E. (1991). *Situated learning: legitimate peripheral participation*. Cambridge: Cambridge University Press.

Mercer, N. (1995). *The Guided Construction of Knowledge*. Clevedon: Multilingual Matters Ltd.

Murphy, E. (2004). Recognising and promoting collaboration in an online asynchronous discussion. *British Journal of Educational Technology, 35*(4), 421 - 431.

Pollard, A., Triggs, P., Broadfoot, P., McNess, E., & Osborn, M. (2000). *What Pupils Say: Changing Policy and Practice in Primary Education*. London: Continuum.

Pratt, N. (2004). *The National Numeracy Strategy: tensions in a supportive framework.* Unpublished PhD, University of Plymouth, Plymouth.

Pratt, N. (2006). *Interactive Maths Teaching in the Primary School*. London: Paul Chapman Publishing.

Pratt, N., & Kelly, P. (2007). Mapping mathematical communities: classrooms, research communities and masterclasses. *For the Learning of Mathematics, 27*(2).

Schoenfeld, A. H. (1996). In Fostering Communities of Inquiry, Must It Matter That the Teacher Knows "The Answer"? *For the Learning of Mathematics, 16*(3), 11-16.

Wenger, E. (1998). *Communities of practice: learning, meaning and identity*. Cambridge: Cambridge University Press.

PUPILS' MATHEMATICAL REASONING EXPRESSED THROUGH GESTURE AND DISCOURSE: A CASE STUDY FROM A SIXTH-GRADE LESSON

Raymond Bjuland, <u>Maria Luiza Cestari</u> & Hans Erik Borgersen

Agder University College, Kristiansand, Norway

This paper reports research that focuses on pupils' reasoning expressed through gesture and discourse while working on a mathematical task in a school lesson. Our aim is to illustrate how two groups of pupils make the transition between two systems of representation, figure and Cartesian diagram. Through detailed analyses of two group dialogues, we reveal that the gesture strategies <u>pointing</u> and <u>sliding</u> are prominent in both groups. These are also related to the discourse strategies <u>comparison</u>, <u>coordination</u> and <u>going to an extreme location</u>. Our study supports the claim that gesture and speech develop simultaneously in the pupils' mathematical reasoning. The gestures stimulate joint attention and reinforce the verbal discussion among the pupils. They also function as connectors between the two systems of semiotic representation and as memory markers.

INTRODUCTION

This paper is related to the project, Learning Communities in Mathematics (LCM). The project was designed at Agder University College (AUC) in Norway[1] in order to build communities of inquiry in which teachers and didacticians develop the teaching and enhance the learning of mathematics (Jaworski, 2004). The theoretical background for the project was presented at Cerme 4 (Cestari et al., 2006). Teachers from eight schools have participated in workshops designed at AUC working together with other teachers and didacticians on mathematical tasks in an inquiry mode. One of these teachers brings ideas from such a workshop into her classroom, transposing a specific task from the workshop to the mathematics lesson.

Based on dialogues from two groups of pupils in a sixth-grade lesson, we focus on their reasoning expressed through gesture and discourse strategies. Edwards (2005) claims that, until recently, researchers have ignored gestures as an important aspect of communication. She suggests that gestures could contribute substantially to the way we both talk and think about mathematics. By presenting a detailed and plausible analysis of the dialogues from two groups, we see how pupils' gestures are closely related to how they express their mathematical reasoning through verbal interactions. We address the following question: What kinds of gesture and discourse strategies do pupils in our study use when solving a task involving two different systems of representation, a figure and a Cartesian diagram?

THEORETICAL FRAMEWORK

Research has lately attempted to better understand how the role of signs and representations is related to mathematical activity and communication (Hoffman et al., 2005). Otte (2005) emphasises that mathematical problems, which are difficult for students to understand, can be more clearly analysed by teachers and researchers from a semiotic perspective. Children learn to operate with signs, and they learn how to use them, which illustrates that signs play a crucial role in their mathematical reasoning. According to Confrey (1995), signs and symbols are used as mediators of sociocultural participation in order to offer educators productive ways to understand the importance of using gesture, imitation, and inflection in the construction of knowledge. In our work, we particularly focus on the pupils' gestures as an important aspect of learning and communication.

More recently, the investigation of gesture in mathematics is situated within a theoretical framework that sees cognition as an embodied phenomenon (Edwards, 2005). In this theory, thinking is embodied at multiple levels: a) Instantaneously, through gaze, gesture, speech and imagery; b) Developmentally, through personal experiences for example with mathematics manipulatives; c) Biologically, capabilities and constraints are developed through evolutionary time (Edwards, op. cit.). According to this author, research into the relationship between language and gesture has added a new dimension to the embodied cognition approach. Speech and gesture contribute with different aspects to communication and cognition. These aspects "are elements of a single integrated process of utterance formation in which there is a synthesis of opposite modes of thoughts" (McNeill, 1992, p. 35). Edwards (2005) claims that gesture can be seen as a bridge between imagery and speech, seeing gesture as a nexus bringing together "action, visualization, memory, language and written inscription" (p. 5).

From an analysis of a dialogue in a fourth-grade classroom, Bartolini Bussi (1998) shows how words and gestures have different but related functions: words refer to the general situation, while gestures refer to the particular one. This is also revealed in Núñes (2004), who was able to show that gestures, related to linguistic expressions, stimulated dynamic thinking in real time among his subjects. Gesture could be defined as "movements of the arms and hands … closely synchronized with the flow of speech" (McNeill, 1992, p.11). Based on this definition, we are aware of the fact that gestures include many aspects that are not addressed in this study.

Mathematical discourse strategies are problem-solving strategies expressed in group discussions. Examples of typical discourse strategies used in solution processes are *posing questions*, *monitoring*, *looking back*, and *visualising* (Bjuland, 2007).

Mathematical representation systems

The three semiotic resources language, symbolism and visual images function together in mathematical discourse but these resources fulfill different functions as

the mathematics text unfolds (O'Halloran, 2005). The functions and resulting grammar for these semiotic resources may be conceptualised as three integrated systems. O'Halloran emphasises that meaning expansions occur when the discourse shifts from one semiotic resource to another.

The use of symbolism as well as pictures and diagrams is fundamental in mathematical thinking. Most textbooks make use of a variety of systems of representation in order to promote understanding (Janvier, 1987). This author emphasises the translation processes which are "the psychological processes involved in going from one mode of representation to another, for example, from an equation to a graph" (p. 27). Behr et al. (1987) focus on five distinct types of representation systems that play an important role in mathematics learning and problem solving: real scripts (real world events), manipulative models (e.g. Cuisenaire rods), static pictures or diagrams, spoken language, and written symbols. These authors also claim that translations among these representations and transformations within those are important.

METHOD

The present study has adopted the dialogical approach to communication and cognition (Cestari, 1997; Linell, in press) as a tool to understand how pupils use semiotic tools when solving tasks. We have chosen this approach to the data analysis since it "allows one to analyse the co-construction of formal language among participants in a defined situation" (Cestari, 1997, p. 41). This approach allows us to identify interactional processes, which, in the analyses of these group dialogues, are the pupils' utterances expressing their discourse strategies used in the solution process. More recently, after using video recordings to collect data, we are also able to identify the pupils' gestures produced during the resolution of the task.

In the analyses of the group dialogues, the pupils' utterances are presented with our comments in brackets, written in italics. In these comments, we have also included the mathematical representations, figure and diagram, illustrating the pupils' focus in one representation and their shift to another one.

Semiotic analysis of the task

Based on research dealing with interpretation of Cartesian diagrams representing situations, Janvier (1987) has inspired Norwegian researchers to focus on different representations of functions (Gjone, 1997). In our study, we focus on one of these tasks that have been used in the KIM project in Norway (Gjone, op. cit.) in which a total of 1900 pupils distributed over grades 5, 7 and 9 were tested on several mathematical topics. We present a semiotic analysis of this task in order to emphasise movements between different mathematical representations.

The pupils were working on the following task: <u>Write down which person corresponds to each of the points in the diagram</u> (the Norwegian words *alder* and *høyde* mean age and height respectively).
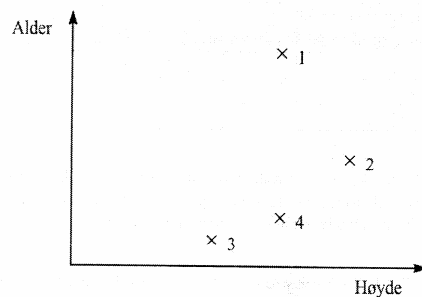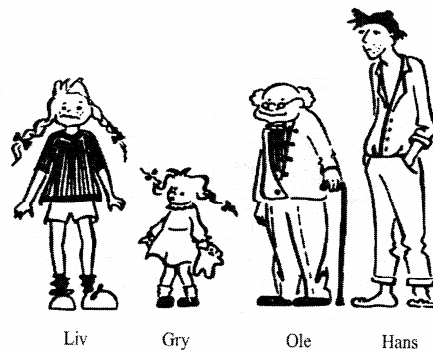
Liv corresponds to point   ………………….

Gry corresponds to point   ………………….

Ole corresponds to point   ………………….

Hans corresponds to point   ………………….

This task was originally introduced by the Shell Centre for Mathematical Education (1985) and comprised seven persons who are to be represented by a corresponding point on a Cartesian diagram. The pupils are particularly challenged to make the transition between two systems of representation, from figurative elements to the Cartesian coordinate system. They are then confronted with the relation between the variables height and age. There are no introductory comments to the task that could have given it a context. The information is given based on the picture of the four persons and the diagram. In contrast, in the original version of the task, there is a context since the seven persons are standing in a queue at a bus stop.

The task illustrates three different semiotic categories. The first one corresponds to a figurative category that is represented by the drawing of four people. The picture shows the different nature of signs, illustrated by different gender, age and height, clothing, use of glasses, stick, long and short hair, etc. The second semiotic category

can be identified as a Cartesian coordinate system with two axes: the vertical one indicating age, and the horizontal indicating height. Both axes have an arrow, showing increasing age and height respectively, without an indication of units. Four points are marked on the diagram with a cross and labelled 1, 2, 3 and 4 respectively. The third semiotic category can be identified as written questions asking the pupils to link each person's name from the figure and the labelling of points in the Cartesian diagram. The pupils are expected to write their answers in a ready written schema.

THE TASK IN USE IN THE CLASSROOM

The total time spent on this task is 19 minutes of one lesson. The mathematical activity is introduced in a plenary section before the pupils work on the task in groups of two or three. After the small-group work, the teacher in a plenary discussion with her pupils sums up and concludes the solution process of the task.

Initially, the teacher introduces the task by using an overhead to give her pupils the opportunity to focus on the figurative image of the four persons. She uses the verb *see* several times, inviting her pupils to be attentive to the visual representation. This conversation gives the pupils a first approach to the relation between the variables height and age. The teacher also contextualises the task by suggesting that the persons have been out for a walk. She then points to the diagram, focusing on the transition from the figurative elements in the picture to the Cartesian diagram by making a connection between people and the labelling of points through gestures.

Approaching the task: three boys

In the dialogue below we illustrate how three boys approach and make sense of the task. Throughout the whole conversation, two of the boys are standing next to each other, looking down at their own sheet of paper showing the task. They do not write anything while they are working before they sum up and come up with a solution. They use their fingers in the solution process as a tool in two ways: to point and to slide from the picture of the four persons to the Cartesian diagram, and to slide within these two kinds of representations.

23	Pupil2:	I think one is (…) (*Pointing at point 1, diagram*).
24	Pupil3:	I don't know.
25	Pupil2:	No but age (*Sliding along vertical axis, diagram)* he is tall, no oldest (*Repeated pointings at Ole, figure*).
26	Pupil3:	Mmm.
27	Pupil1:	If it's not him that's youngest (*His gaze is directed to the task sheet of pupil 2*).
28	Pupil2:	Yes he is tallest up there (*Sliding upwards vertical axis)*.
29	Pupil1:	She is shortest (*Pointing at Gry on his own sheet, figure*).
30	Pupil2:	Yes.
31	Pupil1:	It can't be her.
32	Pupil2:	So she is three (*Pointing and holding at point 3, diagram*).

33	Pupil1	Or four.

| 34 | Pupil2 | No she is three. She is shortest. (*7 sec.*) (*still holding at point 3, totally 14. sec.*). But eh she Gry (*Pointing at Gry, figure*) she is three (*Pointing at point 3, diagram*). She is youngest in age (*Sliding along vertical axis, diagram*) and then she is shortest in height (*Sliding along the horizontal axis*, *diagram*). |

Pupil 2 points to both the diagram (23) and to the picture (25), focusing on Ole as a possible candidate for point 1. He is also sliding along the vertical axis when he introduces the variable of age into the conversation (25), relating the variable to one of the dimensions of the Cartesian coordinate system. The dialogue shows that Pupil 2 is moving between the two systems of representation: from a particular point in the diagram, via the vertical axis of the Cartesian diagram to the picture of Ole. The gesture strategies *pointing* and *sliding* are crucial for this process. Pupil 1 follows up this initiative. It seems as if he is making a *comparison* of Ole and Hans since he focuses on another 'him' that is not Ole (27). Pupil 1 uses the Norwegian word 'minst' in two different meanings: youngest (27) and shortest (29). However, Pupil 1 and Pupil 2 (28) do not seem to have any problems in understanding each other in the conversation.

Pupil 1 is then changing his focus from Hans, *pointing* to Gry (29). Pupil 2 follows up this initiative and relates Gry to point 3 in the diagram (32). He points and holds at point 3 for about 14 seconds, using the pointing as a memory marker. It is interesting to observe how the one-dimensional perspective (height) from Pupil 1 has been elaborated on and moved into a two-dimensional perspective (age and height) by Pupil 2. The pupils use the strategy of *going to an extreme location* that is highlighted as being an important strategy in problem solving.

The wrong suggestion from Pupil 1 (33) provokes Pupil 2 to justify why Gry should be located at point 3 in the diagram. By moving his finger from the picture of the little girl to the point in the diagram, he repeats how the two different systems of representation are related, from the one-dimensional picture to the two-dimensional diagram. In his explanation he also focuses on the two variables, age and height, demonstrating on the axes with his finger that Gry is the youngest and the shortest person in the figure (34). The discourse strategy of *coordination* of the two dimensions is related to two consecutive *pointings* and *slidings* respectively. The reasoning of Pupil 2 illustrates how the two different tools of semiotic mediation, the utterance expressed and the gestures are related and develop simultaneously.

In the continuation of this dialogue, after having considered the extreme location and placed Gry as point 3, the boys argue that Liv corresponds to point 4 since she is older than Gry. The comparison of the females is then followed by a comparison of the males, in which they come up with a solution for Ole and Hans respectively.

Approaching the task: the two girls

The girls have met some obstacles in the solution process. Pupil 4 has suggested that Gry and Liv correspond to point 1 and point 2 respectively, indicating a start from left to right in the figure. She also has indicated a one-dimensional perspective, focusing only on the variable age. Pupil 5 has suggested that Hans corresponds to point 1. One possible explanation could be that Pupil 5 only focuses on the one-dimensional variable, height, indicating a misconception: the tallest person in the picture corresponds to the point, located highest in the diagram.

In the continuation of the dialogue, we give a brief section of the discussion between the girls. They use their pencils for pointing and sliding.

74	Pupil5:	Yes I didn't see this because here is height (*Sliding along the horizontal axis, diagram*) and here is age (*Sliding along the vertical axis, diagram*). No, I saw it now. Gry (*Pointing at Gry, figure*) she is smallest and she is
75	Pupil4:	Number one
76	Pupil5:	Number three (*Pointing at point 3, diagram*).
77	Pupil4:	Three?
78	Pupil5:	Hang on. Hans (*Pointing at Hans, figure*) is tallest. Then he should be placed out there.
79	Pupil4:	Hans is [tallest]
80	Pupil5:	[Age]
81	Pupil4:	But look [if it's height there]
82	Pupil5:	[But look now, tallest] Hans there (*Pointing at point 2, diagram*) and then that man (*Pointing at Ole, figure*), and there Ole (*Pointing at point 1, diagram*).
83	Pupil4:	He is [he is (…) Liv.] (*Pointing at Liv, figure*).
84	Pupil5:	[(…) oldest]
85	Pupil4:	Liv she is taller than him (*Pointing at Liv and Ole, figure*). We go first for the height.
86	Pupil5	Okay. Then it's, but she (*Pointing at Liv, figure*) she is number four (*Pointing at point 4, diagram, 8 sec.*). Gry was there (*Pointing at point 3, diagram*). And then it's Liv (*Marking point 4, written answer*) and Ole (*Marking point 1, written answer*). I think it's like this.

In this dialogue, Pupil 5 applies the gesture of *sliding* along the horizontal and the vertical axes, focusing on the variables height and age respectively (75). This initiative is related to her first idea in which she suggested that Hans corresponded to point 1. Pupil 5 goes on to use the strategy of *going to an extreme location,* focusing on the extreme person, Gry, who is both youngest and shortest.

Pupil 4 sticks to her wrong suggestion (75), but Pupil 5 has discovered, probably from her *sliding* along the axes, that Gry corresponds to point 3 (76). The brief question from pupil 4 (77) provokes an explanation since they have come up with different suggestions for Gry's location. However, Pupil 5 does not seem to be

interested in discussing this at this particular moment in the dialogue since she goes on focusing on Hans (78). Her wrong suggestion (Hans corresponds to point 1) is now tested against her new understanding based on the extreme location of Gry. It is clear that Pupil 5 focuses on both the variable age (80) and the variable height (82). She *points* at Hans in the picture (78) and at point 2 in the diagram (82), showing that she is moving between two systems of representation in a proper way. This is confirmed when she *points* at the correspondence between the figurative element of Ole and point 1 in the diagram (82). She makes it clear that Hans is the tallest person (82) and Ole is the oldest one (84). The discourse strategies of *coordination* and *recapitulation* of the solution for Gry have been stimulated by three consecutive *pointings*, moving from figure to diagram. Pupil 5 points and holds for eight seconds at point 4, finding the solution for Liv. The gesture functions as a memory marker to locate one of the points in the diagram. Pupil 5 also marks the solutions for Liv and Ole (86), indicating that she moves between all the three semiotic representations given in the task.

In the continuation of the dialogue, the students raise their hands to get help from their teacher, and they say that they are stuck. We could ask ourselves why they do this when we have observed that Pupil 5 has found a proper solution. One possible explanation could be that Pupil 5 is not convinced about her solution, and it is also possible that Pupil 4 is distracting her. Throughout the dialogue, we observe that Pupil 4 sticks to the one-dimensional perspective, focusing only on the variable height (79), (81), (85). Another possible explanation could be that the pupils are little attuned to each other's perspective, indicating that they do not seem to have established a mutual relationship.

The girls have two dialogues with their teacher in which they express their difficulties. However, based on the teacher's open questions, they manage to find a solution to the task.

DISCUSSION AND CONCLUSION

Through the detailed analysis of dialogues from two groups of six-grade pupils working on a task in a problem-solving context, we have identified the boys' and the girls' gesture and discourse strategies respectively. A semiotic analysis of the task reveals how it stimulates the pupils to move between different mathematical representations, moving from a concrete, visual picture of four persons to a more abstract data representation of a Cartesian coordinate system. The analyses from both groups have also revealed that the pupils point and slide within one and between two representation modalities, moving their fingers or pencils between figure and diagram. So, in this case *pointing* and *sliding* are the main gestures used to make the passage between figure and diagram.

The analysis of the dialogue in the boys' group illustrates that the discourse strategies, *comparison*, *coordination* and *going to an extreme location* are crucial in

order for them to come up with a solution. They use the strategy of *pointing* and *sliding* followed by a *comparison* of two and two persons of the same sex all the way using *coordination* of the variables height and age. The two girls also come up with a solution by using *pointing* and *sliding* along the coordinate axes. Simultaneously they use the discourse strategies, *comparing* two persons, *going to an extreme location*, *recapitulating* a solution, and making a *coordination* between the two axes. Our examples from the two groups reveal the important relationship between these two mathematical reasoning expressions, indicating that discourse and gesture have different but related functions. This is also supported by the work of Bartolini Bussi, (1998), Edwards (2005) and Núñes (2004).

What is the significance of noticing examples of mathematical reasoning expressed through gesture and discourse? What can we learn from an exploratory study from a school lesson, identifying these strategies used by pupils working on a task that stimulates movements between two data representations? The translation process of going from one mode of representation to another is important for pupils in order to promote their mathematical understanding (Janvier, 1987; Behr et al., 1987). The interplay between gesture and discourse strategies seems to be a mediating device in the pupils' collaborative problem solving. The gestures stimulate joint attention and reinforce the pupils' mathematical reasoning through their speech. They also make connections between the semiotic representations, and they function as memory markers (Edwards, 2005), remembering by holding on a point in the diagram.

According to Núñes (2004), human gesture constitutes the forgotten dimension of thought and language. This author claims that speech and gesture are in reality two facets of the same cognitive linguistic reality. Our study confirms that gesture and speech develop simultaneously in pupils' mathematical reasoning. However, more research taking an embodied approach to cognition for understanding language needs to be carried out in order to learn more about how discourse and gesture are related and how gestures function, in order to study pupils' mathematical reasoning.

NOTE

REFERENCES

Bartolini Bussi, M. G. (1998). Joint activity in mathematics classrooms: A Vygotskian analysis. In F. Seeger, et al. (Eds.), *The culture of the mathematics classroom* (pp. 13-49). Cambridge: Cambridge University Press.

Behr, M., Lesh, R. & Post, T. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representations in the teaching and learning of mathematics* (pp. 33-40). Hillsdale, NJ: Lawrence Erlbaum.

Bjuland, R. (2007). Adult students' reasoning in geometry: Teaching mathematics through collaborative problem solving in teacher education. *The Montana Mathematics Enthusiast*, *4* (1), 1-30.

Cestari, M. L. (1997). *Communication in mathematics classroom. A dialogical approach*, Unpublished doctoral dissertation. Oslo: University of Oslo, Norway.

Cestari, M. L., Daland, E., Eriksen, S., & Jaworski, B. (2006). The role of didactician/researcher working with teachers to promote inquiry practices in developing mathematics learning and teaching. In *Proceedings of the Fourth Conference of the European Society for Research in Mathematics Education* (CERME 4), Saint Feliux, Spain.

Confrey, J. (1995). How compatible are radical constructivism, sociocultural approaches, and social constructivism? In L. P. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 185-225). Hillsdale, NJ: Lawrence Erlbaum.

Edwards, L. (2005). Gesture and mathematical talk: Remembering and problem solving. Paper presented at *American Educational Research Association Annual Meeting*, Montreal, Canada.

Gjone, G. (1997). Kartlegging av matematikkforståing. Rettleiing til funksjonar. Oslo: Nasjonalt læremiddelsenter.

Hoffman, M. H. G., Lenhard, J. & Seeger, F. (2005). *Activity and sign – Grounding mathematics education*. New York, NJ: Springer.

Janvier, C. (1987). Translation processes in mathematics education. In C. Janvier (Ed.), *Problems of representations in the teaching and learning of mathematics* (pp. 27-32). Hillsdale, NJ: Lawrence Erlbaum.

Jaworski, B. (2004). Grappling with complexity: Co-learning in inquiry communities in mathematics teaching development. In *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol 1, pp. 17-36. Bergen, Norway: Bergen University College.

Linell, P. (in press). *Essentials of dialogism. Aspects and elements of a dialogical approach to language, communication and cognition.* http://www.tema.liu.se/

McNeil, D. (1992). *Hand and mind: What gestures reveal about thought.* Chicago: University of Chicago Press.

Núñes, R. (2004). Do real numbers really move? Language, thought, and gesture: the embodied cognitive foundations of mathematics. In F. Iida et al. (Eds.). *Embodied artificial intelligence* (pp. 54-73). New York: Springer.

O'Halloran, K. (2005). *Mathematical discourse. Language, symbolism and visual images.* London: Continuum.

Otte, M. (2005). Mathematics, sign and activity. In M. H. G. Hoffman, et al. (Eds.), *Activity and sign – Grounding mathematics education* (pp. 9-22). New York, NJ: Springer.

Shell Centre for Mathematical Education (1985). *The language of functions and graphs.* Manchester: Richard Bates Ltd.

HOW MATHEMATICAL SIGNS WORK IN A CLASS OF STUDENTS WITH SPECIAL NEEDS: CAN THE INTERPRETATION PROCESS BECOME OPERATIVE?

The case of the multiplication at 7th grade

Isabelle Bloch – IUFM d'Aquitaine – France - isabelle.bloch@univ-pau.fr

Abstract: This paper addresses the use and signification of mathematical signs in the teaching/learning situations we build for students with special needs. We observe that students experience great difficulties within the dynamics of interpretation: their interpretation of signs – as a zero – cannot evolve from the first signification they meet in class. We use C.S. Peirce's theory of semiotics to understand this phenomenon: signification is not definitely deduced from (mathematical) signs because interpretation is a triadic process that requires an interpretant. We give examples of situations that can lead students to see numbers as products and enter the operative way mathematical signs work. These situations involve language interactions – 'interpreting games' – and try to rely on the milieu of the situation.

Keywords: signs, interpretation, Peirce's semiotics, situations, multiplication.

I. PEIRCE'S SEMIOTICS AND SITUATIONS

For about twenty years we have been involved in a research work about mathematics teaching for children having learning difficulties, what is called 'Specialised Teaching' in France. In this research we try to understand how students having failed in their previous studies understand mathematical problems, try to solve them, use mathematical signs and knowledge. Among other phenomena, in these classes we can observe signs being produced, interpreted and used by students in very unusual ways from a mathematical point of view: this contributes to bring the necessary interpretation process to a close, instead of introducing an interpretative evolution.

The theoretical frame we use is due to Brousseau, for the Theory of Didactical Situations, and C.S. Peirce for its semiotic part. According to Saenz-Ludlow (2006), "For Peirce, thought, sign, communication, and meaning-making are inherently connected. (…) Private meanings will be continuously modified and refined eventually to converge towards those conventional meanings already established in the community. (…) "… A whole sign is triadic and constituted by an object, a 'material sign' (*representamen*), and an *interpretant*, the latter being an identity that can put the sign in relation with something – the object. A very important dimension in Peirce's semiotics is that interpretation is a *process*: it evolves through/by new signs, in a chain of interpretation and signs. The *interpretant* – the sign agent, utterer, mediator – modifies the sign according to his/her own interpretation. This dynamics of signs' production and interpretation plays a fundamental role in mathematics where a first signification has always to be re-arranged, re-thought, to fit with new and more complex objects.

Peirce – who was himself a mathematician – organised signs in different categories; briefly said, signs are triadic but they are also of three different kinds. We will strongly resume the complex system of Peirce's classification (ten categories, depending on the nature of each component of the sign, representamen, object, interpretant: see Everaert-Desmedt, 1990; Saenz-Ludlow, 2006) by saying that an *icon* is a sign that refers to the object as itself – like a red object refers to a feeling of red. An *index* is a sign that refers to an object as a proposition: 'this apple is red'. An *symbol* is a sign that contains a rule. In mathematics all signs are symbols to be interpreted as *arguments*, though they are not exactly of the same complexity; and so are the language arguments we use in mathematics for communication, reasoning, teaching and learning. The semiotic theory will help us to identify the kind of sign produced in teaching-learning interactions, and the appropriateness (with regard to the situation) of how students interpret the given signs. Then we use the theory of didactical situations to build situations appropriate to knowledge.

Signs and situations

Mathematics aims at the definition of 'useful' properties, that that can help to solve a problem or to better understand the nature of concepts. A strong characteristic of these properties is their invariance: they apply to wide fields of objects – numbers, functions, geometrical objects, and so on. This implies the necessity of flexibility of mathematical signs and significations. To grasp the generality and invariance of properties, students have to do many comparisons – and mathematical actions – between different objects in different notational systems. While the choice of pertinent symbols and different classes of mathematical objects is necessary to reach this aim, it is not sufficient. To produce knowledge, the situation in which students are immersed is essential. By 'situation', we mean the type of problems students are led to solve and the milieu with which they interact. Brousseau's Theory of Didactical Situations (Brousseau 1997) claims that to make mathematical signs 'full of sense' – which means that signs have a chance to be related to conceptual mathematics objects – it is necessary to organise situations that allow the students to engage with validation, that is, to work with mathematical formulation and statements. In Bloch (2003), we explained how we build situations where the aimed knowledge appears as a condition to be satisfied in a problem. This complex building of situations and signs can be realized to teach multiplication in specialised classes.

II. PRODUCTION AND (MIS)INTERPRETATION OF SIGNS

We present here the teaching design we undertook to deal with the interpretation of mathematical signs in a special-needs-students class (14 to 15 years-old). We first consider how they (failed to) solve multiplication problems. We want to point out some common characteristics of the interpretation in such classes, and try to deduce some principles to build appropriate situations for numeration and multiplication. These situations have been experimented during two academic years with the students of a SEGPA [1], in the south of France.

How students interpret mathematical signs

In every class we can observe some transitional problems regarding students' production and interpretation of signs:

- There exists a kind of *phenomenological entropy*: production of a variety of personal signs, drawings, gesture… (see Brousseau, 1997; Saenz-Ludlow, 2006; Steinbring, 2006). This could actually not be a problem as long as it would be possible to bring the students back to the usual signs in relation to the knowledge.

- Students have an interpretation that sometimes is not connected to the right knowledge: they elaborate personal schemes, procedures… and a formula is sometimes associated to a more ancient knowledge (for instance $17.3 \times 10 = 17.30$). It is a well known phenomenon that can create obstacles.

Especially in classes of students with special needs, these phenomena tend to be permanent and to put obstacles in the way of the construction of knowledge and the progress of didactical time. Moreover, there are some heavy tendencies to 'misinterpretation':

- The sense given to a sign is only the original one, and students tend to 'freeze' the first signification encountered and its contingent manifestations: a zero is a sign of a tenth, so it cannot be the sign of a 'lacking hundred' as in 2.708 for instance.

- A great difficulty lies in the fact that very often mathematics reiterate twice an operation, or put together two different arguments in a chain of interpretation and signs, as we already said. For example, in an integer a zero may be a sign of a tenth, but *two* zero are not a sign of *two* tenths: it is rather the sign of a *square* tenth.

- We can detect a wide-ranging *interpretative deflation*, that is, students interpret mathematical signs as if they were only conventional: a sign is the print the teacher writes, but it has no signification as an *interpretant* of a mathematical object, neither as a tool. It is very usual that students encounter great difficulties in seeing a mathematical sign as including a rule: an argument according to Peirce's theory.

For instance, students having heard of proportionality for at least four years get a numeric table and are required to say if the situation is of proportionality or not. We can see that for some students, the numeric table is really an *argument* (in Peirce's sense, with a rule embodied in it): they are able to say that in such a table, doing some computation you could find the rule – the proportionality coefficient – and even build images of new numbers. For some other students, the table is obviously an *index* of proportionality, that is, they are aware that the table contains indications, and tells them something about proportionality, but they are not able to find a relevant indication in it; and some other students just see the table as an *icon*: the thing the teacher draws on the blackboard each time she speaks of proportionality. This misunderstanding affecting students' interpretation of symbols begins at primary school with numbers and operations: multiplication is a significant example because a lot of numeration skills is needed to solve it.

It is well known that, even in 'ordinary' classes, students still experience difficulties with the multiplication table as far as 7th or 8th grade. At primary school a number as 63 is rather easily seen as 60 + 3 even by a 'weak' student. We say that the included numeration *arguments* are perceived. However it becomes more problematic with big numbers, and the difference between a zero of a tenth and a zero of a hundred. It does not work anymore with products: for students a writing as 9×7 is seen as a product, but lots of them do not know *which* product, because they cannot tell the 'result' of this multiplication: we say that (9) × (7) is the *index* of a product for them (it contains '×') but no more. Moreover, considering 63 as the result of a multiplication is not possible: for most students it is not even the *icon* of a product, students cannot see it this way. This lack of flexibility in the interpretation of numbers entails a heavy handicap in calculation and resolution of problems.

We could think that calculation means as pocket calculators could avoid the necessity of 'learning by heart' usual products, especially by students with difficulties, but even with a calculator you need to know which calculation is required by your problem. The multiplication table is nevertheless an interesting mathematical object to be learned at school because of its usefulness in mental calculation and of its social meaning – ability to solve problems of money for instance. Moreover, the challenge is to enable them to understand better numbers and various ways of writing numbers – and beyond the numbers, what are mathematical objects and signs and how we can operate with them. For this purpose, we tried to implement teaching/learning situations about multiplication in a 7th grade class of SEGPA students. Students in this class have failed to achieve a reasonable knowledge on arithmetic operations in their previous studies.

To introduce students' experience of variety of interpretation in the field of calculation, we organised three stages, including assessment, on multiplication items. Each stage includes situations of validation with regard to a milieu, which the theory of situations points as necessary to make conceptions evolve.

III. THREE STAGES FOR SITUATIONS ABOUT MULTIPLICATION

Numeration, division and multiplication by 10; number of tenth, figure of tenth

First problem: A school wants 3140 tickets for students' meals and you must calculate how many booklets they have to order. Each booklet has ten tickets.

Second problem (the episode in the class will not be described here) students have to determine the number of tenths and the figure of tenth, the number of hundreds and the figure of hundred, in a rather 'big' number, e.g. 3457. 'Real' tickets are available: tenth of tickets have to be put in a white envelope; then tenth of white envelopes are put in a brown envelope; and tenth of brown envelopes are put in a big packet (see Destouesse, 1997). This situation leads to materialize both the tenths and the hundreds, and the number of tickets that lay in an envelope: the signs here are envelopes of tenths or hundreds. Tickets being always available, it is possible to see that there are *really* a hundred of tickets in a hundred...

The third problem is: writing 'big' numbers as 96 708; be able to tell apart the figure of tenths(hundreds) and the number of tenths (hundreds). Didactical variables are the size of the numbers and the existence of a zero or not. We want to observe how students cope with a zero, depending on the place of the zero.

Second stage: the Pythagoras games

First game: lotto. Each student gets 81 cards to play. They are two players, playing at their turn. The teacher puts four cards on the empty table. Players can put a card on the table if, *and only if,* it has got a common edge with a card that already lies on the table. A player wins if he/she puts all his/her cards on the table. If you have got 12 on the table you can put 15 or 16; but if you have got 15 you cannot put 16 because it has just a corner in common with 15. The condition of the common edge is essential because it compels students to justify that they are allowed to put a card.

Second game: the frequencies. As the table is full of all numbers, students must color the numbers in, but there is a rule: each color corresponds to a frequency. Numbers that are once in the table are green; those that are twice are yellow; the numbers that are three times are blue, and four times violet (anyway students themselves choose the colors). To do this task, students must wonder why 12 appears four times; try to find lines and columns where 12 is apparent: the table gives a sign that a number is a product, and a product from a number of ways. At that moment this is a task about decomposition in factors, and no more: calculate a contingent product. Then the teacher asks students to write all the decompositions they can find with the colors; this allows the interpretation of numbers as products. A new rule incorporated in a mathematical sign must be recognised by the students; this stage carries an important contribution to the flexibility of signs.

Assessment stage: count rectangles

To apply the reconstructed knowledge and associate the rules – rule about zeroes, tenth and hundreds, rules about multiplication – students are asked to calculate some products. They have schemas – rectangles of squared papers – and they are encouraged to make decompositions in 10×10. Products are 8×27; 16×25; 32×48; 53×78… They are allowed to use the Pythagoras table for validation of partial products. Global validation of the result is done by a class debate as a synthesis.

## Methodology

The methodology is a clinical observation (as in Saenz-Ludlow, 2006). We observed students at work and made a transcription, and videotapes when possible: sometimes it is problematic (for the students or even the teacher) in this kind of 'special' class.
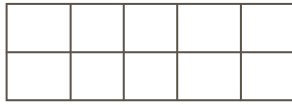
IV. EXPERIMENTATION AND RESULTS

Students' work and production of signs

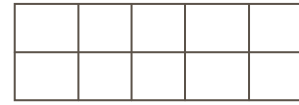*First problem and second problem: numeration*

Some students think there will be more booklets than 3140. Some others try to multiply by 10. We find schemas with a booklet and the numbers to multiply:
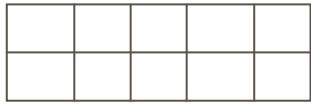
× 4   ~~× 100~~ × 10 *(crossed by student)*   × 300

These drawings demonstrate an rather good understanding of what is expected, but students seem to be unable to calculate or to reason without drawings: they are not able to undertake a solving procedure relying only on numbers. These schemas work as a kind of reminder of what a tenth is (*icon* or *index* of a tenth) and seem to be easier to interpret for the students than 10×10 or 10×300. However some students use writings like: $3140 \div 10$ that evidently comes from a previous encounter with this situation; other ones prefer $3000 + 100 + 40$. The two signs are handled with not equivalent mastery, as the first one "gives the answer" but can remain obscure for some students. The second one is less evident as a solution because there is still one step to do in order to obtain the answer, but this formula gives a better explanation of why it is so: it is an *argument* of decomposition of the number and allows a different kind of validation. These behaviors – writings of numbers and drawings – tend to show that students have learned rules of calculation but the writing of the rule can be disconnected from the signification, signification that is restored in drawings. Drawings and calculation are then a basis for a mathematical debate about writings of integers and signification of zeroes. Thanks to the validation phase, interpretation of booklets and zeroes can evolve: from being first icons of tenths they become symbols in the decomposition of the number. The second situation (Fourmillions) allows then to go deeper in the signification of tenths, hundreds and thousands.

Third problem: In this problem (writing 'big' numbers with zeroes and saying how much tenths or hundreds there are in a number), we can see a student writing $96\,708 = 967 \times 10 + 8$. Another one wants to 'add a zero' but he does not know where. We can observe that for most of them, *a zero works anyway as a sign of a tenth, regardless of its place in the number*: this is the phenomenon we brought up in §II, the signification that have been seen first is frozen: a zero works as an *icon or an index of a tenth*, wherever it is in a number. This is however rather surprising, considering that they just played the two precedent games before (booklets of tickets and Fourmillions): it shows how this knowledge about integers and numeration is problematic and long to be well set up. Signs of numbers, digits, zeroes, have to be explored in a lot of situations before students being able to see all the connections between the different significations.

Anyway this problem proved to be rather difficult because students had only numbers written in digits to work on. They hardly thought of doing drawings and did not refer to the previous situation (Les fourmillions), although this one would have been useful to interpret the figure of a hundred and the number of hundreds.

The Pythagoras games

Students were surprised to discover that 16 does not follow 15, neither 64 follows 63… This game makes students aware of the *structure* of the table, which is not the numeration one as they used to think. The rules determine this structure: in a column you can move forward by adding the number at the top of the column… and it is the same with a line. This provides an *argument* that a player has the right to put a card on the table. The second game (frequencies) induces the necessity of knowing how much times a number appears: how many ways of breaking down an integer into products? This game is a real success: students both perform works of art in coloring, and write factorizations (a work of art in mathematics...).

Assessment phase

In this situation (example 46×37), some students begin with squares of 5×5, but they get discouraged when they see their classmates are more successful doing packs of 10×10.

46

37

40 × 30

6 × 30

7 × 40

6 × 7

Notice that the real dimension of the packs on the drawing is not meaningful, which students perfectly manage: it is a schema that supports reasoning. At this very moment the square pattern works no more as an icon or index of tenths, it starts operating as an argument: that is, the rule is embodied in it and students use the pattern as a schema supporting calculation. This new way of using signs clearly extends to the use of numeric ones, as we can see in the following description.

Students prove to be able to combine the different rules: rules of numeration and multiplicative arguments. Actually they perform calculations that they were unable to do before, such as 30×40: $3 \times 4 = 12$, but there are two zeroes, so $30 \times 40 = 1200$, and they interpret the result in a pertinent way: there are thousand and two hundreds of little squares in one rectangle. Language interactions are numerous, with a dimension of reasoning and validation: "The result *cannot be else* than have a digit '2' at the end because $6 \times 7 = 42$". They use the Pythagoras table as a help; it works now like a formula, what Peirce calls a *hypoicon*: it means that the signification has been embodied in it. A *hypoicon*, also called a diagram, is an argument that has been incorporated: you just do the work with it but without even the necessity of thinking. According to Peirce, all algebra expressions are hypoicons.

This stage appears to be fundamental as it allows summing up the whole knowledge that has been built: the use of the tenths and hundreds, the products, the numeration. The assessment of the device has to be measured at this final stage: if it leads to a success, this is the success of the whole process of restoring the signification of numbers' writings and products as arguments, and the flexibility of signs.

Semiotic analysis of the student's work

- We can notice that the number of rules that are embodied in an argument as '0' is still a big problem for the students; a long time is necessary to make them able to discriminate the right signification in each situation.

- Even more than in 'ordinary' classes, students work with private rules, commonly linked with a previous context: students are very sensitive to the first context, and *décontextualisation* remains difficult. The first meaning they encounter is 'frozen'.

- These characteristics make especially difficult to work about *dynamics* of interpretation; yet this dynamic is an important part of the essence of mathematics itself, and getting an intelligence of dynamics is necessary to allow students going further with the learning of mathematics and conceptualisation.

- Once arguments have been embodied, they work correctly as *hypoicons* as expert mathematicians use them; this is an important result since it proves that students have become able to use the knowledge in a correct mathematical way.

- As also in 'ordinary' classes, a major difficulty is to be noticed about the inversion of arguments: divide by ten is not seen as the inverse of multiply by ten; combination of rules are difficult too, such as: divide (multiply) by hundred is divide (multiply) by ten and once more by ten. We also know that up to secondary school, it is very problematic to understand that $(a/b)/c = a/(bc)$, but even more the inverse.

As we first thought, we could observe that playing the situations:

- tenths and hundreds can be restored as entities but also as 'containing' quantities of units; it means that signs could get the evolutional dimension and the plasticity they lacked until this moment;

- signs evolve from icons or indexes to symbols and finally the Pythagoras table turns into a *hypoicon* of multiplication rules as it is suitable for a mathematical use.

Persistent phenomena in the didactical contract of 'weak' classes

However, in such 'special' classes, we notice that some difficulties remain whatever the situations proposed could be. First, the algorithmic level is always difficult to interpret and to be managed by the teacher: when a student only gives a standard answer, the teacher can hardly discern if he/she really knows or if he/she has been trained to this result before (or both!). This is an additional reason to organise such situations: they allow students to produce their own signs, to free themselves from the first signification they could not escape until this moment, and to understand new connections and meanings. By the same time, these situations allow (and compel…) the teacher to observe personal procedures of students and to organise pertinent interventions to make them progress; they also allow validation with a tool (the table, the calculation, a debate on the mathematical truth).

Other specific contract phenomena cannot be avoided, even in this kind of work: for example, in 'weak' classes, the teacher very often anticipates that students will fail. J.M. Favre (Favre, 2003) speaks of "the three failures in special classes: the previous failure – students are here because they have failed in primary classes; the actual failure, that

is objectively not so important but sometimes the teacher 'cheats' by proposing very plain work or avoiding to recognise a failure; the anticipated failure – the teacher always thinks that students will fail, and she carefully avoids too difficult tasks." The teacher actually anticipated that the assessment situation would be too difficult and the students would fail. This was not the case; students managed it very well.

As a paradox, in such a situation the teacher sometimes does not leave useful tools available for her students: in the fourth phase (rectangles) she considered that the Pythagoras table had not to be authorised, though students just used it as a *hypoicon* as already said (and not like a pocket calculator!). This teacher's behaviour is to be linked with a third phenomenon: her pressure on students that they have to give explanation for everything they said or wrote. This is another specificity of the didactical contract in 'weak' classes, but it puts students in a very uncomfortable situation, even as they are more fragile than others.

CONCLUSION

We notice that this experimentation allowed us to achieve global success at the final stage: all students succeeded in counting little squares in rectangles, and they performed an utilisation of the table as we aimed at, that is, as a *hypoicon* with a rule embodied in it. Students have enlarged both their conceptions of numbers: they could now see them as products, and their ability in doing multiplication and understanding interlinked rules of numeration.

We eventually think that the peircean semiotics proves to be complementary with the Theory of Didactical Situations, as it helps making a diagnostic of students' semiotic but also conceptual difficulties. It provides useful indications for building situations and analysing students' work and productions. Students with special needs showed they had a partial and inadequate perception of signs: not only the products were misinterpreted, but still the zero, the tenths and hundreds. Moreover, this signs' misinterpretations go together with phenomenological and conceptual non-flexibility; the situations we build provide validation, which is needed to interpret signs as mathematical ones, but also a variety of signs and utilisations. Then these situations allow students to improve their semiotic flexibility and involve in the process of knowledge. Dynamics of mathematical interpretation can be restored.

NOTES

[1] Section d'Enseignement Général et Professionnel Adapté : students with cognitive difficulties but no disabilities and being behind at least two years. In France such classes are called 'special classes'.

REFERENCES

Bloch I.: 2005, 'Conceptualization through semiotic tools in teaching/learning situations', Communication CERME 4, Barcelona.

Bloch, I.: 2003a, 'Teaching functions in a graphic milieu: What forms of knowledge enable students to conjecture and prove?' ESM 52: 3-28.

Bloch, I.: 2003b, 'Contrats, milieux, représentations dans l'adaptation scolaire', Actes du séminaire national de didactique des mathématiques, 171-186, Paris: Université Paris VII.

Brousseau, G.: 1997, 'Theory of Didactical Situations in Mathematics', Kluwer.

Destouesse, C.: 1997, 'Ça fourmillionne', Grand N 59, p.11-18, IREM de Grenoble.

Everaert-Desmedt, N.: 1990, 'Le processus interprétatif. Introduction à la sémiotique de C.S. Pierce'. Liège : Mardaga.

Favre J.M. (2004) Etude des effets de deux contraintes didactiques sur l'enseignement de la multiplication dans une classe d'enseignement spécialisé. Actes du séminaire national 2003 de didactique des mathématiques, pp. 109-126, Paris : IREM Paris VII.

Otte, M. : 2006, 'Mathematical epistemology from a peircean semiotic point of view', ESM, 61.

Peirce, C.S.: 1898, 'Reasoning and the Logic of Things', The Cambridge Conferences. Republished 1992, Harvard University Press.

Peirce, C.S.: 1897 to 1910, 'Collected Papers' , Republished 1931-1958 Harvard University Press.

Radford, L.: 2002, 'The seen, the spoken and the written: A semiotic approach to the problem of objectivation of mathematical knowledge', For the Learning of Mathematics, 22-2, 14-23.

Radford, L.: 2004, 'Rescuing perception: Diagrams in Peirce's theory of cognitive activity', Research Program, Research Council of Canada.

Sackur, C.: 2000, 'Experiencing the necessity of the mathematical statements', Proceedings of PME 24, Hiroshima, Japan, IV 105-112.

Saenz-Ludlow, A.: 2006, 'Classroom interpreting games with an illustration', ESM, 61.

Steinbring, H.: 2006, 'What makes a sign a mathematical sign? An epistemological perspective on mathematical interaction', Educational Studies in Mathematics, 61.

Annex: the frequencies

	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

ANALYZING THE CONSTRUCTIVE FUNCTION OF NATURAL LANGUAGE IN CLASSROOM DISCUSSIONS

Paolo Boero, Valeria Consogno

Dipartimento di Matematica, Università di Genova

The aim of this paper is to identify some mechanisms through which social interaction results in knowledge construction and reasoning development in mathematics. Previous analyses had put into evidence a peculiar function of natural language in classroom discussions as a tool to transform and develop the content of the discourse through interactive mechanisms of linguistic expansion based on key-expressions. The aim of this paper is to go in-depth in the analysis of such mechanisms. In particular, three mechanisms will be described; experimental evidence will be provided about their functions in the development of mathematical discourse in the classroom.

THEORETICAL PERSPECTIVE, AND PURPOSE

This contribution belongs to the streams of research that deal with the constructive function of natural language in mathematics.

In the last decades, an important trend of research in mathematics education has been the increasing attention paid to language and semiotics aspects in the construction of mathematical knowledge, both in an individual and in a social construction perspective. This occurred in relationship with research advances in other domains (psychology, linguistics, hermeneutics). Let us consider the perspective of the "constitutive character" of natural language (see Bruner, 1986, Chapt.4): on one side, it suggested to consider whether other semiotic systems (in particular, algebraic language) share the same potential, and how students can approach the "mathematical realities" inherent in the specific expressions of those systems (cf Sfard, 1997, and Radford, 2003); on the other side, it opened the way to study how the "mathematical realities" are "constituted" during verbal activities in the classroom (cf Sfard, 2002).

With reference to these streams of research, we can take into account previous studies that are related to the issues considered in this report. Boero (2001) and Consogno (2005) consider how mathematicians deal with algebraic or natural language written expressions. In the case of algebraic expressions (Boero, 2001) crucial steps of a mathematician's activity consist in the reading of the algebraic expressions produced by him/her: sometimes this reading suggests ideas that go far beyond what the reader thought during the writing phase. The novelty can consist in the discovery of a possibility to simplify the expression, in the discovery of a new meaning, or in the anticipation of some moves that can allow to achieve the goal of the activity. In the case of natural language expressions, Consogno (2005) considers the flow of the writing/reading phases during individual activities of conjecturing and proving performed by undergraduate mathematics students. The *Semantic-Transformational*

Function (STF) of natural language is the construct that accounts for some advances of their conjecturing and proving process. The student produces a written text with an intention he/she is aware of; then he/she reads what he/she has produced. His/her interpretation (suggested by *key expressions* of the written text) can result in a *linguistic expansion* and in a *transformation of the content* of the text that allow advances in the conjecturing and proving process.

Douek (1999) is concerned with the analysis of the role of argumentation during classroom discussions aimed at the construction of mathematical concepts in activities of elementary mathematical modelling of physical phenomena. She identifies *lines of argumentation* whose development and crossing contribute to the enrichment of concepts both in terms of reference situations, operational invariants, linguistic representations (according to Vergnaud's definition of concept: Vergnaud, 1990), and in terms of maturation towards the level of *scientific concepts* (Vygotsky, 1990, Chapter VI). The analyses show how a *line of argumentation* in some cases develops through someone's interpretation of linguistic expressions produced by some others, far beyond their intention in producing them.

The aim of the study reported in Consogno, Boero and Gazzolo (2006) was to see if the STF of natural language (see Consogno, 2005) can account for the development of a *line of argumentation* during a classroom discussion (by focusing on those phases when oral productions by some students are interpreted by other students), and how it works. That report addressed two questions:

I) Can classroom social construction of mathematical meaning be interpreted in terms of STF (i.e. of semantic *transformations* that happen through *linguistic expansions* produced by someone, and suggested by *key expressions* uttered by some others)?

II) Can a student profit, in classroom discussion, by others' interventions (in order to develop his/her intuitions) through mechanisms that involve the linguistic transformations of his/her own expressions?

The reported research not only allowed to answer those questions in the specific case of a long term construction of probabilistic thinking in a primary school class (from grade I to grade IV), but also raised further questions, summarised in the following excerpt:

> The voice of a student can provoke (through specific key expressions) an interpretation by another student related to his/her perception of the evoked situation, that comes back to the first student as an enrichment or a transformation of his/her original intuition (…). In that case the other student plays a role that could be interiorised through a mechanism of inner dialogue supported by a written text (like in the episodes analysed by Consogno, 2005). In other situations a chain development happens: different students can play complementary roles to transform the situation under consideration. It may happen that focus moves from a situation to the opposite situation (…); or that two complementary interventions open the way to the consideration of the whole range of possibilities between the two evoked (…). In the last case social construction of knowledge seem to reveal its highest potential. According to these considerations,

focussing on the STF of natural language seems to offer the researcher the possibility of classifying different patterns of social construction of knowledge in terms of different mechanisms of linguistic expansion. This suggests the need of characterising the variety of "linguistic expansions" that are of interest in the perspective of the STF of natural language."

The aim of the present paper is to better focus on the three mechanisms of verbal interaction briefly presented in the previous text by better describing them and providing further experimental evidence about their role in the social development of mathematical discourse in the classroom. In particular, in this paper we will consider both social construction of mathematical concepts and social construction of mathematical reasoning, thus widening the scope of the investigation concerning the role of STF in social interaction.

METHODOLOGY

Keeping into account the analyses reported in Consogno, Boero and Gazzolo (2006), we will propose a description of the three kinds of mechanisms, which emerged in the previous case study. The descriptions will put into evidence some features that allow to recognize those mechanisms in classroom social interactions, and their functions in the development of classroom discourse.

The following step will be to analyse some "salient episodes" in which those mechanisms have played a crucial role in the development of classroom discussions.

Those "salient episodes" have been identified by considering two teaching experiments:

- the long term teaching experiment on the development of probabilistic thinking from Grade I to Grade IV, reported in Consogno, Boero and Gazzolo (2006);

- a teaching experiment concerning the approach to mathematical proof in twelve classes in Grade VI.

In both cases, audio-recordings of all classroom discussions were available. In both cases, the teaching experiments have been performed by teachers belonging to the Genoa research team in Mathematics Education. Their style of teaching is strongly influenced by their belonging to our research team; in particular, social construction of knowledge according to the model of "Mathematical Discussion" orchestrated by the teacher (see Bartolini Bussi, 1996) is a crucial educational choice in their classrooms.

By "salient episode" we mean a fragment of a classroom discussion in which students make a substantial progress according to the a-priori analysis of the task (see Consogno, Boero & Gazzolo for a detailed presentation of criteria to choose "salient episodes" in the case of the first teaching experiment).

The analyses of the "salient episodes" will have a double aim: to provide experimental evidence for the relevance of the three mechanisms in the social construction of knowledge and in the development of mathematical reasoning; and to show how their functioning can be interpreted in terms of the STF of natural language.

The analysis of the "salient episodes" will be performed according to a modelling perspective: students' utterances will be interpreted "as if" their thinking processes would fit the models of reasoning proposed by us. This is a legitimate choice until students' words do not explicitly contradict our interpretation. Naturally, different and equally coherent interpretations might be possible in some cases.

THREE KINDS OF MECHANISMS OF DEVELOPMENT OF CLASSROOM DISCOURSE

I. Evolution of a personal interpretation of the situation

One student's interpretation of the problem situation is enriched and integrated by the interventions of some schoolmates who propose other interpretation(s) of the same situation, up to the full apprehension by the first student and his/her relevant contribution to the solution of the problem in the classroom discussion.

II. From a situation to the opposite one, to a wider perspective

Students' contributions put on the table two opposite situations related to the task (for instance, one case fits the conditions of the task, while the other escapes them). This contributes to construction of knowledge by offering a wider perspective for a discourse that embraces both cases and allows a jump in conceptual construction and reasoning.

III. From single cases to generalisation

Students propose some similar cases related to the task, then a collective process of induction takes place by considering common features of the evoked cases. A general statement is the outcome of the process.

…AND CLASSROOM DEVELOPMENT OF REASONING

The examples have been taken from transcripts concerning the following task proposed to twelve classes of junior high school (Grade VI in Italy) at the end of the year, in the perspective of developing mathematical reasoning and approaching mathematical proof:

To prove in general that two consecutive numbers have only 1 as their common divisor

The educational aim of the task was to offer an occasion to move from justification based on examples, to general argument concerning "*whatever numbers*". Indeed the empirical search for divisors of consecutive numbers soon becomes heavy and boring, thus "*reasoning in general*" can become (under the teacher's guide) a shared opportunity in the classroom.

In the a-priori analysis of the task we had considered the possibility of two different strategies (one based on the consideration of remainders, the other based on the consideration of the distance between two consecutive multiples of the same number). It was expected that this possibility might have offered an opportunity to compare and share different ways of reasoning to solve the problem.

The research aim of the task was to analyse different ways of social construction of knowledge. Indeed the variety of possible strategies, the shared need for general arguments and the complexity of linguistic and mathematical aspects inherent in the task offered an opportunity to observe how different, personal verbal contributions (and ways of thinking) might "converge" in the social construction of a solution. For instance, we can say that *a number is a divisor of another number if it divides it; or if the remainder of the division of the latter number by the former one is zero*. These different ways of speaking about the divisibility of one number by another correspond to different ways of thinking about that concept, thus offering the students different opportunities to approach the solution of the problem.

Mechanism I

Paolo: A number is a divisor of another number…it means that it divides it…A number divides another number when it is contained exactly a certain number of times in it, nothing remains (*non resta niente*, in Italian) in the dividend. Now I have two consecutive numbers… A number and the following number… Nothing remains in the previous number… (long silence)

Lucia: A number is divisible by another number when the remainder is zero (il resto è zero, in Italian). If I move to the following number… the remainder… (long silence)

Paolo: The following number is the previous number increased by one… Thus the remainder is one… If I divide the following number by a divisor of the previous number, I get one as remainder, so the following number is not divisible by that divisor.

In this case, the verb "*remains*" ("*resta*" in Italian) uttered by Paolo suggests the noun "*remainder*" ("*resto*" in Italian) to Lucia, while the expression "*the following number…the remainder*" uttered by Lucia suggests to Paolo both the "*increased by one*" and "*the remainder one*" (a crucial linguistic expansion in order to get a full apprehension of the problem situation). Then Paolo can conclude his reasoning by considering the remainders (zero, i.e. divisibility; one, i.e. non divisibility) of the division of two consecutive numbers by a divisor of the first number. Note that the transition from the verb *"remains"* to the noun *"remainder"* (i.e. from an "inclusion" to a "division" point of view) performed by Lucia allows Paolo to enter the more familiar situation of "remainders" of divisions, which students were widely accustomed to in previous grades.

Mechanism II

Rosy: In case of divisibility, the remainder is zero

Lorena: While in case of non divisibility, the remainder cannot be zero

Daniele: In the case of two consecutive numbers… (long silence)

Francesca: In the case of one number and the following one… (long silence)

Ivan: In the case of the following number, we move from remainder zero to remainder one, so the following number is not divisible by that divisor

"*In case of… the remainder*" is the key expression that allows moving from a situation to the opposite one, and then to a linguistic expansion that embraces both cases and allows to finalize reasoning. Note also how Francesca contributes to the debate by transforming the expression "*Two consecutive numbers*" (coming from the task) into the expression "*one number and the following one*", which allows Ivan to "see" the transition from "*remainder zero*" to "*remainder one*".

Mechanism III

Maria: In the case of two as divisor, we need to move from one even number to the following one, two steps far.

Barbara: While in the case of three as divisor, we need to move from a divisible number to the next number divisible by three… three steps far

Francesco: And in the case of four, four steps far!

Lorena: The distance is growing more and more, when the divisor increases… the distance is the divisor! … (long silence)

Roberto: So if the distance is one, the only divisor is 1.

The expression "*… steps far*" ("*…passi distante*" in Italian) allows students to move from one example to another, then the idea of "*distance*" ("*distanza*" in Italian) allows to embrace all the examples in a general statement that Roberto can particularise in the case of interest for the problem situation. Note that in the Italian language students can move easily from the adjective "*distante*" to the noun "*distance*".

Compound processes

In some cases we have observed composition of different kind of social construction of mathematical reasoning, like in this example, where a process of type III contributes to a process of type I:

Elena: One number and the following one… an even number is followed by an odd number, this means that 2 cannot be a common divisor…It would be a common divisor for the following one, four … I must make a jump… (long silence)

Fabio: It is necessary to make a jump of two places

Stefania: If one number is divisible by three, the following number that is divisible by three is three places farther…

Gina: And four places farther in the case of a number divisible by four…

> Elena: I understand: if one number is divisible by another number, then the following case of divisibility will be as far as the divisor!

Elena considers even/odd numbers, probably (if alone) she would have not been able to leave that situation. Fabio "sees" the jump of two positions, and Stefania and Gina suggest further examples that expand the range of exploration. Finally Elena realizes that "*two places farther*", "*three places farther*", "*four places farther*" can bring to "*as far as the divisor*".

In this complex social construction, we can see how the expression *"make a jump"* uttered by Elena suggests to Fabio the expansion "*make a jump of two places*", a new interpretation of the same fact evoked by Elena. "*A jump of two places*" suggests to Stefania the linguistic transformation "*Three places farther*", which allows Gina to produce another example "*four places farther*". Elena integrates those contributions in a more general statement that links to her initial interpretation of the situation.

… AND SOCIAL CONSTRUCTION OF KNOWLEDGE

For the first mechanism, we will consider a *salient episode* of construction of knowledge taken from the teaching experiment on the development of probabilistic thinking from grade I to grade IV, presented in Consogno, Boero & Gazzolo (2006).

We will consider another salient episode from the same teaching experiment, where mechanisms of type II and III intervene.

For further details concerning the teaching experiment, the a-priori analysis allowing to identify the *salient episodes*, etc., and the analysis of a third episode related to a mechanism of type II, see the same Report.

Shortly, the episodes were taken from transcripts of audio recordings collected during a long term construction of probabilistic thinking (from Grade I to Grade IV) in one primary school class. The episodes concerned two "jumps" in the evolution of probabilistic thinking, corresponding to relevant conceptual acquisition by students.

Mechanism I

Grade III: in couples, students will throw two dice; they will bet on *odd* or *even* according to the number got by adding the digits of the dice. Before playing the game, the question is: *"Is it better to bet on odd or even?"*. Individual answers follow, and a discussion takes place. At the beginning of the discussion, students consider odd outcomes: 3, 5, 7, 9, 11; and even outcomes: 2, 4, 6, 8, 10, 12. *Even* seems more likely to come out because the number of even outcomes is bigger. But…

> Elisa: I agree with Mattia, as he considers the results.
>
> Giulia: Mattia has considered all possibilities, because he has considered the two dice and has put the results and (I think) has looked at all possibilities
>
> Teacher: Is it the same thing to think of the result or to think of the two dice?
>
> Mattia: It is the same thing … no… yes!

Giulia:	If you think of dice… to the digit shown by your dice… because the result is one digit plus another digit that makes a result. Before adding them, those two numbers are alone, they are not together… because if one casts 3 and the other 4
Roberto:	for instance, 4 is a number and 3 is another number, as Giulia told, if you add them, they make 7, but before putting them together, 4 is a solitary number and 3 is another solitary number, then when they go together we get a number formed by smaller numbers
Giulia:	yes, but before getting the result, the two numbers can be other numbers.

The teacher asks Giulia to make an example, then she invites the other students to produce other combinations. The way is open to consider all possible equally likely outcomes. *NOTE that in Italian the "digits" of the dice are called "numbers".*

In the reported fragment Giulia re-elaborates the distinction (suggested by the teacher) between the dice and the result in terms of numbers: the addends and the results. The intervention of Roberto not only echoes Giulia's intervention, but expands it and suggests a transformation of the content (putting into evidence, by saying *" for instance "*, the fact that the couple 3 and 4 is an example; and the fact that the sum is a *"number formed by smaller numbers"*). Note how, thanks to the syntactic construction, "*a number formed by smaller numbers*" opens the possibility to see a number as formed by smaller numbers in different ways. Then Giulia is able to see how those *"smaller numbers"* can be different from 3 and 4. We can interpret *"a number formed by smaller numbers"* as the key expression that suggests a linguistic expansion that results in a semantic transformation of the original idea of Giulia. The interpretation of the situation by Roberto comes back to Giulia as an opportunity to enrich her way of thinking and contribute to the advancement of classroom discourse.

Mechanisms II and III

Grade III: students approach the idea of ratio between the number of favourable cases and the number of all cases as a measure of probability of an event. Students are asked to make a choice between two games: the game with a coin (by betting on heads or tails), or the game with a dice (by betting on one of its digits). Giulia writes:

> In the case of the coin there are much more possibilities. For instance, suppose that in two labyrinths there are 2 paths (in the former) and 6 paths (in the latter). The 2-paths labyrinth offers more possibilities to get out, if in each labyrinth there is only one exit.

The text produced by Giulia is chosen by the teacher to feed a classroom discussion, because it can help the students to compare on the same, neutral ground (labyrinths) two different random situations. Note that in the text produced by Giulia (as well as in all the other texts) there is no trace of reasoning in terms of "ratio" *("more possibilities"* concerns only the comparison of 5 against 1). Yet no work on the ratio concept had been performed before this episode. Note also that Giulia orients the discussion towards a simplified, yet abstract model of labyrinth. The teacher asks to

take position on Giulia's text and to evaluate if her last sentence (*"The two paths labyrinth…"*) was necessary, or could have been omitted.

Anna: I agree with Giulia that in the 2-paths labyrinth you get out earlier, in the case of the 6-paths labyrinth you must try all the paths and you spend a lot of time.

Matteo: But in the 6-ways labyrinth you do not need to try all the paths, because for instance the first time you fail the exit, but then at the second or third trial you may find the good way to escape… You don't need to try all the paths!

Giovanni: It is necessary to consider the condition posed by Giulia, namely that there is only one exit, otherwise all the paths might have an exit, and it would not be a labyrinth any more!

Mattia: If a labyrinth would have more exits than paths with no exit, practically it would be very easy to escape, on the contrary if the labyrinth has the same number of paths and exits, …it would be easier but the exits must be more than one half of the number of the paths.

Some voices: less than one half!

Teacher: I would like Mattia to repeats his sentence - please, listen to him, then we will discuss what he said

Mattia: Can I make an example? In the 2-paths labyrinth there is one exit, while in the 6-paths labyrinth there are 3 exits; in order to make the 6-paths labyrinth easier than the 2-path labyrinth, you must put exits to more than one half paths, because if in the other labyrinth there are two paths and one exit, it is one half.

Anna's and Matteo's considerations suggest to Giovanni the reason why the condition posed by Giulia is necessary: the expression *"all the paths"* (be it necessary to try all of them, or not) can suggest the fact that if *"all the paths have an exit"* then it is sure that one can escape from the "labyrinth" in each trial. The situation is transformed by passage to a non-labyrinth limit situation. Then the extreme cases of one exit and six exits opens the way to Mattia to consider the number of exits in the 6-paths labyrinth as a variable that can take values between one and six. He tries to express the idea that the right comparison with the 2-paths situation must be made by considering *"one half of the number of the paths"* as the discriminating case. In terms of the STF, he performs some linguistic expansions (*"more exits than paths with no exit (…) the same number of paths and exits"* in his first intervention, and then *"an exit to more than one half paths"* in his second intervention) of the limit situations uttered by his schoolmates, which results in a transformation of the situation: the number of exits becomes a variable related to the number of paths.

By this way Mattia moves from the set of cases proposed by his schoolmates to a general consideration of the relationships between the number of exits and the number of paths. From Mattia's second intervention on, several more and more precise interventions will concern *"3 exits out of 6", "one half of the exits",* and so on, till to the explicit comparison between 3 out of 6 and 1 out of 2 as *"one half"* in both cases.

DISCUSSION

The analyses of some salient episodes, belonging to different teaching experiments, show how the three mechanisms of social development of classroom discourse can fit (as descriptive models) what happened in the classrooms, and how the STF model can account for the functioning of those mechanisms (as an interpretative model).

Further directions of research are suggested by the performed analyses: to identify other mechanisms (if any) of social development of classroom discourse; and to investigate the educational conditions (didactical contract, shared values in the classroom, etc.) that allow the mechanisms described in this paper to work. In particular, listening to the others, freely using (and transforming) the schoolmates' productions, and sharing the aim of solving the problem situation as a collective enterprise seem three necessary conditions.

REFERENCES

Bartolini Bussi, M.:1996, 'Mathematical Discussion and Perspective Drawing in Primary School', *Educational Studies in Mathematics*, 31, 11-41.

Boero, P.: 2001, 'Transformation and Anticipation as Key Processes in Algebraic Problem Solving', in R. Sutherland et al. (Eds), *Perspectives on School Algebra,* pp. 99-119. Kluwer, Dordrecht.

Bruner, J.: 1986, *Actual Minds, Possible Worlds*. Harvard University Press, Cambridge, Ma.

Consogno, V.: 2005, 'The Semantic-Transformational Function of Everyday Language', M. Bosch (Ed.), *Proceedings of CERME-4*. St.Feliu de Guixols.

Consogno, V.; Boero, P.; Gazzolo, T.: 2006, 'Developing Probability Thinking in Primary School: A Case Study on the Constgructive Role of Natural Language in Classroom Discussions, *Proceedings of PME-XXX*, Vol. 2, pp. 353-360, Prague.

Douek, N.: 1999, 'Argumentation and Conceptualisation in Context : A Case Study on Sun Shadows in Primary School', *Educational Studies in Mathematics,* 39, 89-110.

Radford, L.: 2003, 'Gestures, Speech and the Sprouting of Signs', *Mathematical Thinking and Learning*, 5(1), 37-50.

Sfard, A.: 1997, 'Framing in Mathematical Discourse', *Proc. of PME-XXI,* Vol. 4, pp. 144-151. Lahti.

Sfard, A.: 2002, There is more to discourse than meets the ears: Looking at thinking as communicating (...). *Educational Studies in Mathematics*, 46, 13-57.

Vergnaud, G.: 1990, 'La théorie des champs conceptuels', *Recherches en Didactique des Mathématiques*, 10, 133-170.

Vygotsky, L.S.:1934, 1990, *Pensiero e linguaggio*. A cura di P. Mecacci. Bari: Laterza.

ASSESSMENT IN THE MATHEMATICS CLASSROOM. STUDIES OF INTERACTION BETWEEN TEACHER AND PUPIL USING A MULTIMODAL APPROACH

Lisa Björklund Boistrup

Stockholm Institute of Education/Stockholm University

Several researchers stress the fact that students focus their learning according to the content in the assessment and to how this is carried out. Assessment in this particular study is not in a "usual" formal situation. Instead it refers to assessment which can be found in the interaction between teacher and pupil during hands-on work in mathematics. The analytical tools are derived from research of formative assessment, the National syllabus in mathematics and a multimodal approach within a social semiotic frame. The results indicate that pupils do not always get constructive feedback when showing meaning-making in mathematics. Possible reasons for this are discussed from an institutional perspective.

BACKGROUND AND RESEARCH FOCUS

In this paper assessment is considered as a concept with broad boundaries and by this there is assessment going on, explicit or implicit, during every lesson in mathematics. Examples of what can be part of assessment are diagnoses that teachers give to pupils, documentation such as portfolios, feedback in classroom work etc. When a teacher approaches pupils who are working on mathematical tasks, parts of the teacher's communication with the pupils are based on some kind of assessment. In this paper the focus is on the feedback processes between teacher and pupil. I also, finally, discuss possible explanations for the teachers' actions in this particular case from an institutional perspective.

What is assessed and how the assessment is carried out influence pupils' learning (see e.g. Black & Wiliam, 1998; Gipps, 1994). Black & Wiliam (1998) analysed several (250) studies, all of which have formative assessment in focus. Many of the studies show, among other things, that it is important that pupils get feedback on what qualities their performances show and also on what they should focus their learning on in the near future. The studies that Black & Wiliam have analysed rely on quantitative methods. In fact, they stress the importance of qualitative studies for the field of assessment. The lesson with hands-on work, described in this study, is part of a project in which the teachers and researchers worked collaboratively to explore possible meanings of qualities of knowledge/abilities the pupils are expected to develop according to the national mathematics curriculum. In this particular project the teachers worked in pairs performing lessons which were planned by the teachers and researchers in collaboration. The pupils in the study are 10 years old.

The purpose of this study is to find out more about the assessment that takes place in a classroom during experimental work about measurement and volume. The research questions are (The words in italics are described on p 3-4):

A. What is the mathematical focus of the interaction in relation to the feedback processes during the hands-on work of measurement and volume? – *Ideational meaning*.

B. What different (communication) modes do the teachers show (or not show) acknowledgement of in the feedback during the interaction with the pupils? – *Textual meaning.*

C. What kind of feedback is taking place between teachers and pupils during the work within a mathematical frame? – *Interpersonal meaning.*

FRAMEWORK – FORMATIVE ASSESSMENT, GOALS AND MULTIMODALITY

The basis for this study is: (1) research of formative assessment with the importance of feedback (Black & Wiliam, 1998) as discussed above; (2) "goals to aim for" in the national curriculum in mathematics (Swedish National Agency for Education, 2001), because the teachers in the project were supposed to let these goals inform classroom work; and (3) a multimodal approach within social semiotics, mainly how it is described by Kress et al. (2001).

Goals to aim for

There are a total of 14 "goals to aim for" in the national curriculum. The goals that are most relevant to this study are:

The school in its teaching of mathematics should aim to ensure that students

- develop an interest in mathematics, as well as confidence in their own thinking and their own ability to learn and use mathematics in different situations

- appreciate the value of and use mathematical forms of expression

The aim should also be that students develop their numerical and spatial understanding, as well as their ability to understand and use:

- different methods, measuring systems and instruments to compare, estimate and determine the size of important orders of magnitude (Swedish National Agency for Education, 2001, p 23-24).

A multimodal approach

This multimodal approach emphasizes that learning can be seen in a social semiotic frame and that communication is considered not only from a linguistic perspective; instead all modes of communication are recognised. Modes can be, for example, speech, writing, gestures and pictures. Each mode has its "affordances" in relation to the specific situation and people engaged in a communication (Kress et al., 2001),

that is which mode is "chosen" in a specific situation is not arbitrary; instead it is the best way for this person to communicate in this particular moment.

From a social semiotic perspective there are three kinds of meaning that all communication is understood to reflect; ideational, textual and interpersonal. In Morgan (2006) these functions are used with a focus on linguistics and on the construction of the nature of school mathematical activity. The origin of the three functions is from Halliday but in this paper I am using them with a focus on multimodality according to Kress et al. (2001) and with a focus on assessment in mathematics. *Ideational* meaning can reflect what is going on in the world. *Textual* meaning refers to formation of whole entities which are communicatively meaningful and *interpersonal* meaning's focus is on interactions and relations between people. Another feature for this multimodal approach is *signs of meaning-making*. The feature of meaning-making provides, as I see it, possibilities for assessment. Even though a pupil's answer is mathematically incorrect it can be a sign of meaning-making. According to this a teacher does not have to give feedback to a pupil that an answer is incorrect, but instead (s)he can see the opportunity to look at the answer as a starting point for a mathematical exploration. That is, an answer which is "wrong" according to the mathematical discourse can still be seen as a sign of meaning-making of a pupil, and by this as a part of the learning process.

In mathematics education the issue of different forms of representation is not new and in research in mathematics use of different modes is necessary. Often mathematics researchers choose symbols to express mathematical ideas, but use of figurative expressions as in graphs is also quite common, and one can also find written text in comments. However, for some pupils the typical "language" used in mathematics can be one (of several) obstacle to overcome. Lennerstad (2002), Høines (2001) and many others claim that an important issue in mathematics education is to overcome these obstacles by using many forms of representation when teaching mathematics. For assessment in mathematics in Sweden there has been some focus on different forms of representation/expression. One example is a material for formative assessment in mathematics, Assessment Scheme for Analysis of Mathematics (distributed in year 2000 by the National Agency of Education), in which teachers are encouraged to capture their pupils' knowledge in different forms of expression: actions, figures, words, symbols. (Skolverket, 2000b).

ANALYTICAL TOOLS

Following Kress I believe that important aspects of assessment in the interaction are possible to reveal using this multimodal approach.

The *ideational meaning* contributes to the analyses of the mathematical content in the interaction. The content that I am looking for can be derived from the goal about "different methods, measuring systems and instruments to compare, estimate and determine the size of important orders of magnitude". I look for what *signs of*

meaning-making the pupils show in mathematics, especially measurement and volume. I also look for what signs of pupils' meaning-making the teachers (do not) reflect in their feedback. These aspects constitute the analytical base for answering the question about the mathematical focus of the feedback.

Different modes have different "affordances" in interaction according to this multimodal approach. How the modes are used in the interaction is a part of the *textual meaning*. This goes well with the goal to "appreciate the value of and use mathematical forms of expression" (Swedish National Agency for Education, 2001). Still, this seems a little too narrow for this study and this goal is therefore combined with the quote under the headline Assessment in Mathematics: "An important aspect of the knowing is the pupil's ability to express her/his thoughts verbally and in written text with help from the mathematical symbol language and with support from concrete material and pictures" (Skolverket, 2001a). These aspects constitute the analytical base for answering the question about what different (communication) modes the teachers show acknowledgement of in the feedback.

Interpersonal meaning is part of what I look for when dealing with the issue of feedback in general. This fits well with the goal concerned with "develop[ing] an interest in mathematics, as well as confidence in their own thinking and their own ability to learn and use mathematics in different situations" (Swedish National Agency for Education, 2001). In some of the studies referred to in Black & Wiliam (1998) there is evidence that feedback to a pupil on the knowledge she/he has shown in certain tasks has impact on interest and self confidence, whereas feedback on what has to be learned has impact on the learning. As I choose to see it in this study there is some kind of feedback between teachers and pupils taking place every time there is an interaction between them. Pupils are working with a task and the teacher comes by and whether the teacher says something or not she is monitoring the pupils work and makes some kind of assessment. In different modes the teacher shows signs of assessment and the pupils can react to this feedback in different ways. These aspects constitute the analytical base for answering the question about what kind of feedback is taking place between teachers and pupils during the work within a mathematical frame.

METHOD – VIDEO RECORDING

The focus of the data collection is on the interaction between the teachers and the work performed by one group in the class. One video camera is fixed on the group most of the time. The group also has a portable voice recorder on the table.

For the analysis I choose the parts of the films, where the teachers and pupils interact within a mathematical frame and each of these parts are recognized as an "episode". I make multimodal transcriptions of the episodes. Methods for this are described in Rostvall & West (2005) and Kress et al. (2001). I write what each person says – pupils in one column and teachers in another. I also describe their gestures, as well as

their body positions and gaze, in separate columns. The teachers are referred to as T1 and T2. The girls in the pupils' group are referred to as G1 and G2 and the boy as B. Example of transcript:

Time	Speech (Pupils)	Speech (Teachers)	Gestures (S)	Gestures (T)	Body and gaze (S)	Body and gaze (T)	Notes
	The lesson starts with a teacher introduction. The pupils are divided into different groups. Each group gets different measurement instruments, like measuring cups and scales. The group in this sequence got rulers and measuring-tapes.						
1:26	G1- This is two centimetres. This is two centimetres long		G1 holds one piece of pasta and shows it to the rest of the group. G2 puts her hands together.		G1 leans forward and looks first at the piece of pasta and then at the other girl in the group. G2 looks at G1. B looks around and then at the piece of pasta in G1's hand.		Episode 1 starts
1:29	B-Which one?		B is putting the measuring-tape in order.		B looks at the piece of pasta. G2 looks at G1		
1:31	G1-This		G1 takes the pasta piece in her hand and shows it again.		G1 and B look at the pasta. G2 looks at the sound recorder.		
1:33	B1-Is it two millimetres?		G1 puts the piece of pasta on the desk.	T1 has her hands on her hips.	G1 looks at B, then T1 and then at the piece of pasta.	T1 approaches the group and is standing in an upright position. Gaze in direction to the group table. Smiling?	

ANALYSES OF ONE EPISODE FROM THE LESSON

As mentioned in the transcript, the lesson starts with a teacher introduction. The pupils are divided into groups. Each group gets different measurement instruments, like measuring cups and scales. The group in this study gets rulers and measuring-tapes. All groups get pasta (penne) and they receive the task to use their instruments to figure out how much pasta they have got.

All analyses are discussed with and validated by two researchers and also by the two teachers in the study. The analysis of the ideational meaning focuses on what signs of meaning-making the pupils show when it comes to mathematics and how this is (not) reflected in the teachers' feedback. I also articulate which different modes the teachers show acknowledgement of; the textual meaning. The analysis of the interpersonal meaning focuses on to what extent the teacher give feedback. I do write down "all" feedback from the teachers that is not taking place, but this does not mean that my opinion is that this kind of feedback should take place at each occasion. I just want to make visible what is taking place and what is not, with respect to the feedback. For the transcriptions and analysis I have chosen episodes. Each episode starts just before any of the teachers arrive to the group and ends just after the teacher(s) leave(s) the group. In this paper I describe one episode thoroughly. The transcript is divided in parts and each part follows by a short description. After the episode I present an analysis of the episode.

Example of episode

Before this episode starts the pupils try for a while to find a way to use the rulers and measurement-tape. After some time they instead start to count the pieces of pasta:

Time	Speech (Pupils)	Speech (Teachers)	Gestures (S)	Gestures (T)	Body and gaze (S)	Body and gaze (T)
5:50	B-Shall I count?		B has the hands on the table holding on to some pieces of pasta.		G1 and G2 are looking at a girl from another group.	
5:55	G1-We are going to divide all in tens. Here are ten.		G1 shows the groups of tens that she has formed on the table. G2 moves one of the 10-groups in front of G1. Then puts her hands back to the pasta pieces in front of her self. B takes pieces of pasta one at the time and puts them in front of him.		G1 looks at the pasta groups in front of her on the table. G2 and B are looking at her/his hands	
5:59	"All three are counting"		The three pupils are counting groups of tens. Neither of them touches the measuring instruments.			

What we can see here is that the pupils have put the measuring instruments away and they work together putting the pasta into groups of tens. Soon one of the teachers approaches (T1 is standing partly in the way of the camera):

Time	Speech (Pupils)	Speech (Teachers)	Gestures (S)	Gestures (T)	Body and gaze (S)	Body and gaze (T)
6:15		T1-Hey. When you are doing like this, do you have any use for the things you got from T2?	G1 contiues counting.	T1 holds her hands together, then points at the table.	B and G1 are looking at the table.	T1 is standing in front of the table. T2 is standing beside her.
6:18	G2?-No					
6:20		T1-No, was it any point for you getting the things from T2 from the beginning? How could one use them?	G1 contiues counting.	T1 holds her hands in front of her.	B and G1 are looking at the table. G2 is looking at T1.	
	G1?-We don't know.			""		

In the beginning of the interaction two of the pupils do not look at the teacher. Instead they look at the table and on what they are doing with their hands. The teacher goes on pursuing the use of the measurement instruments:

Time	Speech (Pupils)	Speech (Teachers)	Gestures (S)	Gestures (T)	Body and gaze (S)	Body and gaze (T)
6:30		T1-When do you have use for this then?		T1 holds something from the table in her hand and shows it to the students.	G2 looks at T1. G1 looks at G2. B looks at T1's hands.	T1 looks at G2.
6:33	G2-When I am going to measure				G2 looks at T1. B looks at G2 and T1. G1 looks at T1.	T1 looks at G2. T2 leaves the group.
6:36		T1-What do you measure then?				
6:38	G2-The length of something					
6:42	B-Are we supposed to measure every piece of pasta?				B, G1 and G2 look at T1	T1 looks at the group.
6:43		T1-Do you think that they are about the same	T1 points at the pasta.			T1 looks at the group and at the pasta.
6:45	G1-These are two centimetres, this I have already measured		G1 holds a piece of pasta and shows it to T1.		G1 looks at T1 and the piece of pasta. B looks at the table and at T1.	T1 looks at the group.

The teacher tries to make the pupils to find a use for the measuring instruments. She asks them when they usually use these instruments. Finally she makes a suggestion:

Time	Speech (Pupils)	Speech (Teachers)	Gestures (S)	Gestures (T)	Body and gaze (S)	Body and gaze (T)
6:48		T1-Can you put them together, or? I believe that one of you was on to something like that before, who put them a little like this.		T1 puts pieces of pasta together.	All three lean forward and look at the pasta and the teacher's hands.	T1 looks at the pasta.
6:56	G1-I put hem like this beside a ruler		G1 takes a ruler and shows what she means in front of her on the table.		B looks at G1:s hands.	T1 looks at G1's hands.
6:57	B-Yes G1, this can hardly be two centimetres		B takes one piece of pasta in his hand and holds it up in front of him and G1. G1 continues her work on the table.	T1 holds her hands in front of her.	B looks at G1 and the piece of pasta.	T1 looks at B's hand.
7:00	G1-Don't interrupt, I put hem like this beside a ruler		G1 shows how she measured the piece of pasta. B is still holding his pasta piece in the air.		G1 is looking at what she is doing on the table. B is looking at his pasta piece and into the camera.	T1 is looking at the table.
7:06		T1-But try that then, but with this, take this, you place it here on the table		T1 takes a measuring tape in her hand.		T1 is looking at the measuring tape.
7:10	#-We have already done that					
	unhearable					T1 takes a step backwards, puts her hands on her hips and then leaves the group.

Soon after her suggestion the teacher leaves the group and the pupils continues the work on their own:

Time	Speech (Pupils)	Speech (T)	Gestures (S)	Gestures (T)	Body and gaze (S)	Body and gaze (T)
	#-what, okey					
	#-I see				G2 leans her head in her hand. She looks troubled.	
7:18	B-Here, give me that		B takes another ruler in his hand. G2 takes the measuring tape in her hand. She shrugs her shoulders.		G2 says something unhearable to G1 and looks at her. B looks at the rulers and then at the teacher (like he wants her attention).	
7:23	G1-Can't you go and get another instrument, this was hard		B points in the teacher's direction with one of the rulers and starts pushing the pasta together in the table with the two rulers. G1 is touching the pasta in front of her. G2 holds a measuring tape in both her hands in front of her.	T1's hands and arms are freely moving.	G1 looks at the passing teacher and then at the table. She looks troubled. G2 looks at G1 and then at the passing teacher.	T1 laughes and passes the group with her front in direction to the group. Before turning to another group she looks at G2. Now her facial expression is more serious but still smiling.
7:29	B-It is just to do like this		B has two rulers in his hand and pushes the pasta together into a string. G1 holds her hand upon a pile of pasta in front of her. G2 holds her hands closely together in front of her and keeps the measuring tape in her hands.		B looks at the pasta in front of him. G1 and G2 look at B.	

| 7:32 | G2-But how will we know how much everything is? | | G2 stretches out the measuring tape and puts it on the table. | | G2 looks at B and then at the table. | |
| 7:36 | B-But check it out, you take it like this and do little | | B pushes the pasta together into a string with the two rulers. G1 has her hands on the pasta in front of her. G2 has her hands on the measuring tape on the table in front of her. | | G2 looks at the pasta in front of G1. B looks at the rulers that he is working with. G looks first at the pasta in front of her and then at what B is doing. | |

First the pupils still do not know what to do. The teacher passes and one of the girls shouts to her that they want other instruments. The teacher laughs but does not stop. The boy starts pushing the pasta into strings with two rulers and he shows that he can measure the string with the measuring instrument. The girls are looking at what B is doing and after this all three of them do the same thing.

In the analysis for each episode I focus on the three questions, which correspond to the three functions. This is the analysis of this episode:

A. Ideational meaning: The pupils are showing how they make meaning in relation to numbers and also problem solving, when they are putting the pasta pieces into groups of tens. This could, in fact, be seen as a kind of measurement. However, none of this shown knowledge does the teacher show acknowledgement of. Her interest is focused on measurement with the use of the instruments.

B. Textual meaning: The teacher approaches the group and with a quick glance at the table and at the pupils' gestures, bodies and gazes she seems to become aware of what they are doing. The pupils do not from the beginning seem to focus on what the teacher is talking about. They answer her but they are still looking at the table and one of them continues counting. The affordance of gestures is present when the teacher shows something on the table. All pupils in the group lean forward and look at the teacher's hands.

C. Interpersonal meaning: In this episode the teacher gives explicit feedback on what the pupils "should" do next in the mode of speech to the group. She does not give any positive feedback on the signs of meaning-making that the pupils show when she approaches. Later in the interaction she recalls an earlier event and gives feedback on what happened before in the group, what signs of meaning-making she saw then. Most of the feedback is focused on what the group is supposed to do (as opposed to what the group might be learning) and that the teacher might expect the pupils to manage to go through with the task (her laugh).

SUMMARY OF THE WHOLE LESSON

Looking at the lesson as a whole the pattern follows the episode above. In the end of the lesson it is clear that the teachers' intent with the lesson is measuring volume but this is, as I see it, not obvious to the pupils in the class. The mathematical content that is present in the teachers' actions is, most of the time, the use of the measuring

instruments. The focus that the teachers show is more about *doing measurement in a certain way* than *investigating different possibilities to measure (in this case pasta)*. When the pupils show meaning-making which is not included in the teachers' plan for the lesson the teachers do not acknowledge this. It is clear that different modes have different affordances according to the people involved and to the situation, and both teachers and pupils are communicating via speech, gestures etc. The teachers acknowledge modes as gestures in most occasions, but not all the time. At the end of the lesson the pupils in the group have finally solved the task in a way that the rest of the class appreciates (they measure "strings" of pasta and come to the answer 2 meters and 44 centimetres). However, the teachers' feedback is focused on that this method took a long time and was troublesome.

DISCUSSION AND POSSIBLE EXPLANATIONS

When studying the classroom communication in these situations, using the multimodal approach, I find many incidents of formative assessment – that is communication that can be expected to, or at least have the potential to, contribute to the forming of the pupils' mathematical knowledge. Multimodal transcriptions are time-consuming, but do really reveal important aspects of the assessment interaction in mathematics.

A conclusion of the analyses is that the teachers' most important aim of the lesson is the advantages that can be found when using volume instruments to measure (in this case pasta). According to this aim all the teachers' actions are understandable. Throughout the lesson their feedback goes in this direction, so the teachers' actions are very consistent. Their aim with the lesson becomes highly apparent in the end of the lesson when the whole class is gathered. They point out that measuring pasta with a ruler is time consuming and troublesome (despite the fact that the pupils in the class find the method preferable) whereas they point out that other methods are easy to handle (despite the fact that one group using measuring cups find it time consuming and difficult). The teachers are very focused at their aim but on the other hand they stress neither the goal of interest and confidence, nor the general goal of measurement. The pupils followed in this study are really doing what they are told, showing meaning-making during the work, and after a lot of effort they succeed measuring the pasta with the ruler and measuring-tape. Nevertheless the teachers do not, as I see it, give much constructive feedback, which could provide the pupils with possibilities to build on their interest and confidence in mathematics. Constructive feedback on measuring in general would give opportunities for more learning about measurement. Maybe this, the issue about the different goals, is a main point? There are many goals in the syllabus to follow in the teaching and it can be hard for the teachers to capture several of them at the same time. Another question is in what ways the discourses the teachers are part of when it comes to school mathematics affect their teaching and what meaning-making they "capture".

An issue that arises when looking at the results from an institutional perspective (Rostwall & West, 2005) (which is quite close to "context of culture" discussed by Morgan (2006)) is about the collaborative project in which the teachers participated. I start to wonder how much collaboration the teachers have experienced during the project. Maybe the lesson in this study is a lesson which the teachers do not feel familiar with? Maybe they are trying to copy a lesson plan, which they do not grasp fully? If this is the case the teachers' actions are even more understandable. It also points at important issues concerning in-service and collaborative projects with teachers in general, namely issues of cooperative learning for researchers, teachers and pupils and also issues of respect for the teachers' professionalism and for the pupils' contributions to the lessons in mathematics.

REFERENCES

Black, P. and Wiliam, D.: 1998, 'Assessment and Classroom Learning', in *Assessment in Education,* Vol 5, No. 1, 7-74.

Gipps, C, V.: 1994, *Beyond Testing. Towards a theory of educational assessment,* The Falmer Press, London.

Høines Johnsen, M.: 2002, *Matematik som språk. Verksamhetsteoretiska perspektiv,* Liber, Malmö.

Kress, G., Jewitt, C., Ogborn, J. and Tsatsarelis, C.: 2001, *Multimodal teaching and learning*, Continuum, London.

Lennerstad, H.: 2002, 'Matematikutbildning och matematikens två komponenter: innehåll och språk', in *Sveriges Matematikersamfunds medlemsblad.*

Morgan, C.: 2006, 'What does social semiotics have to offer mathematics education research?' in *Educational studies in mathematics,* Vol 61 Nos. 1-2.

Rostvall, A-L. and West, T.: 2005, 'Theoretical and methodological perspectives on designing video studies of interaction', In *International Journal of Qualitative Methods 4.*

Skolverket.: 2000a, *Kursplaner och betygskriterier.* Retrived at http://www3.skolverket.se/ki03/front.aspx?sprak=SV&ar=0607&infotyp=23&skol form=11&id=3873&extraId=2087 060926

Skolverket.: 2000b, *Analysschema i matematik – för åren före skolår 6*, Skolverket, Stockholm.

Swedish National Agency of Education.: 2001, *Compulsory school. Syllabuses*, Retrieved at http://www3.skolverket.se/ki/eng/comp.pdf/ 060824

CERTAINTY AND UNCERTAINTY AS ATTITUDES FOR STUDENTS' PARTICIPATION IN MATHEMATICAL CLASSROOM INTERACTION

Birgit Brandt

Johann Wolfgang Goethe-Universität, Frankfurt a. Main, Germany

In this paper I will present my approach to the collaborative structuring of classroom discourse. The main objective is the active part of children as learners in the ongoing development of the subject matter. Therefore, I use the decomposition of the speaker roles, which was carried out by Brandt and Krummheuer (Krummheuer and Brandt, 2001; Brandt, 2002). In addition to this interactional approach to students' participation, I refer to the model of "certainty and uncertainty" of Huber and Roth (1999) as general individual learning attitudes. The conjunction of these concepts allows us to clarify aspects of the individual participation of several students as well as the coherence of the course of events.

INTRODUCTION

The source of this paper is the collaboration with Götz Krummheuer in a project about argumentation in primary mathematics classrooms. The project was carried out in two Berlin primary schools (supported by the German research foundation; Krummheuer and Brandt, 2001). We examined classroom discourse within the paradigm of symbolic interactionism (Mead, 1934; Blumer, 1969). As I have argued (Brandt, 2004 and 2006), this approach loses the focus on the individual learners by analysing the course of events. The focus is the joint creation of the interaction, whereas the individual responsibility of unique learners for this process can be seen as an expectation of research (Kovalainen and Kumpulainen, 2005, p.247). Thus, my focus is the individual learner as the final "learning instance" (Sutter, 1994, p. 92), even though the ongoing interaction is the social learning condition, which is moulded jointly. Sutter conceptualised this notion of individual learning by participating in (culturally formed) interaction as the idea of "interactional constructivism" (Sutter, 1994). Within this notion of learning-as-participation, an active leaner

− constructs the individual cognition in interdependency with the other learners and bounded to culturally formed cognition (Bruner, 1990),

− co-produces the situational structure of the interaction process bounded to habitual interaction patterns, which are situationally re-constructed by all participants (Bruner, 1983; Voigt, 1995), and

− thereby forms their own learning opportunities as well as the opportunities of the other children joining the learning situation (Naujok, 2000; Brandt, 2004).

With regard to constructivism*, active learners* are often associated with spirited children, taking an active part in an immediate way and challenging the discourse

with new ideas. Then again, a smoothly ongoing classroom discourse needs, at the same time, children participating actively in the opposing way: *non-active* as speaker but as *active recipient* and/or speakers assisting the ideas of others. The participation model carried out by Brandt and Krummheuer (Brandt 2002; Krummheuer 2007) traces these different participation forms with regard to the emerging of the whole event, but without regard to the individual responsibility of an individual learner for this process. Focussing on several individual children, I worked out different participation profiles with different occurrences of activity and engagement.

These different forms of activity can be classified concerning the dimension of certainty and uncertainty for the general learning attitude of an individual learner. Huber and Roth (1999) differentiated students learning attitudes as "seeking" and "finding" (p. 46). They named the more extroverted style as "finding", which means that these students look for new ideas or solutions: They are orientated towards development. "Seeking" is the more security-orientated style, concerning certainty. These students prefer modifications of former solutions, so they are geared towards reinforcing cognition and approved patterns of solution. Elaborating the participation profiles of two first graders in the same classroom, I will demonstrate these different orientations and their impacts on the individual learning process as well as on the ongoing discourse.

THE ANALYSIS METHODS

With regard to the interactional theory of learning mathematics, our exploratory focus is the naturalistic classroom situation with its own dynamics, independence and consistency; the "situational" structuring (Goffman, 1974, p. 8) of the interaction process, which includes the alternating of the active speakers and the interweaved emergence of the subject matter (cf. ATS and SPS, Erickson, 1982; Voigt, 1995). Thus, we videotaped the lessons without any input to the teachers involved, to try to catch everyday occurrences as well as possible. In our past research, we redeveloped and modified several steps of analysis, which had to be applied in order to reconstruct realisations of these everyday situations. Therefore, we analysed detailed transcripts, which contained vocal utterances, physical actions, gestures, and facial expressions of the participants.

With regard to our research, we could identify five dimensions in the precondition structure for the everyday situations in mathematics classes which can be subordinate to our types of analyses (Krummheuer and Brandt, 2001; Krummheuer 2007):

I.   Analysis of Interaction (AI):

−Evolvement of the topic

−Patterns of interaction

−Recipient design

II.   Analysis of Argumentation (AA)

−Analytical structure of processes of explanation and justification

III.   Analysis of Participation (AP)

−Active participation in such processes (production design)

The Analysis of Interaction (I) is the obligatory foundation for further steps of analysis and was developed in the working group of Heinrich Bauersfeld during the 1980's with respect to the ethnomethodological conversation analysis (cf. Voigt, 1984). The Argumentation Analysis (II) is based on Toulmin's (1958) categories for argumentation (for details, see Krummheuer 1995, 2007; cf. Knipping, 2004). According to Miller (1986), we understand learning mathematics as argumentative learning, which means that the participation in argumentations is a pre-condition for the possibility to learn and not only the desired outcome. Mathematical learning in this sense is based on the students' participation in an accountable practice, which we outline by the participation analysis (III). In this paper, I want to emphasise the participation of several students; thus I will dwell on the production design for the active speaking component of participation. We conceptualised the production design through speaker roles with different responsibility. The main idea is that a speaker can have different responsibilities for the current voiced utterance (Goffman 1981). With regard to Levinson (1988), we modified this approach in a more linguistic way. So we deconstructed an utterance into its idea or content and its formulation (see Tables 1 and 2). A speaker can be responsible for the idea and the formulation of the voiced utterance; thus the utterance places new content-related information into the interaction process. But a speaker can also access former utterances; e.g. he/she can support the idea by quoting or reformulating the former utterance. In our former publications, we linked the idea of the utterance to the argumentative function, carried out by the argumentation analysis (Brandt 2002; Krummheuer 2007). Here, I will elaborate the idea of an utterance in a more general way, not for lack of space, but for discussing the possibilities for broadening the application of this analysis method.

Participation Analysis: The Production Design of Utterances

All sequences in this paper originate from a single lesson in a first grade classroom (6-8 year-old children). I will demonstrate the participation analysis using the beginning of this lesson. The numbers from ten to twenty are the subject matter, in particular the quantity aspect of these numbers. As part of the mathematics classroom culture, a string of twenty pearls is a physical object for symbolising the quantity of the numbers between 0 and 20. The string of pearls consists of ten black pearls and ten pale pearls (●●●●●●●●●●○○○○○○○○○○). The children in this classroom know that the string is a 'mathematical object' and they are used to the handling of it, but not all are familiar with it in detail. So, the teacher opens the mathematics lesson by holding up her (bigger) string:

92	Teacher	yehes, now I'm keen to see what the children say *holds a string of pearls in the air:* ●●●○○○○○○○○○○
94	Marina	I see
95	Franzi	*very audible* thirteen.
96		*Marina, Franzi, Jarek and Wayne raise their hands; some children count while whispering; some restate thirteen with a low voice; by and by, more children raise their hands (...)*

In the following, my focus is the participation of Marina and Jarek, two of the pupils who know from the beginning what is going on [96]. Marina directly expresses that she is familiar with the string of pearls [94]. Obviously, Franzi can identify the quantity of the pearls without counting them pearl by pearl but by using the coloured partition. This is not common in the classroom; most of the children must count pearl by pearl. Franzi is the first pupil to answer the question, so we designate her as an author of her own utterance (see Table 1). This means she is responsible for the idea (the quantity of pearls is asked for) and the formulation (just the word 13 for the specific quantity) of her utterance. But her response is not called upon and not accepted as an official answer to the question, which is marked by the raised hands in [96] (this is an aspect of recipient's design):[i]

100	Teacher	*whispering* two three four five fingers I see *counting slowly* six seven eight *louder* Wayne
101	Wayne	thirteen *(Marina and some other children put their hands down)*
102	Teacher	or
103	Pupil	*amazed* uhm
104	Teacher	Jarek
105	Jarek	uhm three plus ten
106	Teacher	or *(Marina raises her hand)* Marina
107	Marina	ten plus three

The teacher waits until a sufficient number of pupils have found the answer, so she ignores Franzi's answer. Then she calls Wayne to answer and he repeats thirteen [95]. His utterance offers no new information to the emerging content of the interaction process, although it is possible that he worked out the solution on his own. Thus, we designate him a relayer (see Table 1). His response was requested by the teacher [100] and it is officially accepted as the answer. So some children put their hands down, including Marina. But the teacher asks for additional answers and it seems that Jarek is not surprised by that enlargement, while other pupils voice astonishment. By formulating the addition three plus ten [105], he presents a new view of the thirteen pearls; he is an author, too. His solution depends on the coloured partition of the pearls. So expressing the coloured partition as addition can be seen as the idea of his response. Now, Marina puts up her hand again. It seems that Jarek's answer gives her a new idea for an additional answer. After being asked for an answer, she offers ten plus three [107] as a different view, but this is directly based upon Jarek's solution: She just reverses the summands, so she finds a new formulation for the partition of 13 pearls. Thus, she supports Jarek's idea and we designate her as a spokesman [see

Table 1]. The teacher accepts this solution as a new one, also the next addition *eleven plus two*. Then, the teacher asks Jarek again:

111	Teacher	or Jarek
112	Jarek	seven minus zero
113	Teacher	*inquiring* seven minus zero
114	Pupil	huh *(other pupils join in)*

Jarek's solution seven minus zero [112] is astonishing and it is difficult to understand for outsiders (like the researchers) how this answer fits with thirteen pearls. The pupils' cries [114f] suggest surprise in the real interaction situation, too. Nevertheless, it is a new idea which does not fit with the former solutions (partitions as additional terms), so he is an author again. He presents his new idea very self-confidently as a claim (as a conclusion in Toulmin's terms). The teacher repeats his response, but she reformulates it as a question. Querying the answer seven minus zero, her question contains a new idea. So she is not a relayer (like Wayne). We designate her as a ghostee (see Table 1), a speaker who uses the formulation of a former utterance for her own new idea. In classroom interaction, this is a typical speaker role for the teacher, just as in this situation. Here is the schema of the different speaker roles:

	Responsibility for the *idea* of their own utterance	Responsibility for the *formulation* of their own utterance	Examples from the transcript above
author	+	+	Franzi [95], Jarek [105, 112]
relayer	-	-	Wayne [101]
ghostee	+	-	teacher [113]
spokesman	-	+	Marina [107]

Table 1: production design – speaking person (the designations in Table 1 and Table 2 are adopted from Levinson, 1988, p. 172)

To complete the production design of utterances, a differentiation of roles for the former speaker with responsibility for the current utterance is appropriate. For example, Marina's response [107] and the teacher's question [113] are linked to Jarek's utterance. Marina supports his first idea [105] with her own formulation, so Jarek is responsible for the idea of the later utterance. However, the teacher contradicts his second idea [112] by repeating Jarek's formulation – but expressing a new idea. In this utterance, he is responsible for the formulation. Thus, both forms of responsibility are different. Following Levinson (1988), we designated these forms as sponsor in the first case and as ghostor in the second one (see Table 2). Wayne cites a response of Franzi; thus, she is responsible for the idea and the formulation; she is the deviser (see Table 2) of Wayne's response:

	Responsibility for the *idea* of the actual utterance	Responsibility for the *formulation* of the actual utterance	Examples from the transcript above
deviser	+	+	Franzi [101] (for Wayne)
sponsor	+	-	Jarek [107] (for Marina)
ghostor	-	+	Jarek [113] (for the teacher)

Table 2: production design – non-speaking person with responsibility (cf. Levinson, 1988, p. 172)

The Responsibility of Individual Speakers for the Development of the Topic

This differentiation demonstrates the responsibility of several persons for the emerging topic; for example, it is possible to differentiate between more teacher-dominated discourses and more pupil-dominated discourses by the length of utterances. Focussing on a single person, it is possible to trace the influence of this person in the classroom discourse without guessing about the intentions of this person, but just by the interactional effects of the utterances. This will be demonstrated by Jarek's participation in the further development of the lesson:

116	Teacher	let's try it, come to the front, seven minus zero *(short break)* Jarek has said something and we have to check that… come here
118	Jarek	*goes to the front*
119	Teacher	*holds up a string of pearls in Jarek's direction* show us seven minus zero show us seven turn around to the class so that the children can see it and so that everyone can compare *holds her own string of pearls up again still showing* ●●●○○○○○○○○○○ so seven
122	Jarek	*silently counts the pearls on his string in the front of the classroom*
123	Teacher	count aloud
124	Jarek	*counts out on his string holding it up in the air* one two three four five six seven holding the string: ●●●●●●● minus zero *he drops down the end of the string he counted; shows* ●●●○○○○○○○○○○ is thirteen

In this sequence, Jarek demonstrates his idea *seven minus zero is 13 pearls* as an author. This idea is controlled by his idiosyncratic handling of the string – perhaps as a kind of cyclic object. By using the hidden black pearls (seven) for his calculation, it is obvious that he has a good knowledge of the quantitative features of the pearls. As in the offering of his solution [112], he seems very self-confident in his demonstration. Subsequently, the teacher repeats his demonstration (again as ghostee) and works out that his handling is not the allowed way of using the string of pearls. She works out that his handling refers to the term 20 - 7 (see Brandt and Krummheuer 2001). Her action confirms that the string is only an embodiment of the quantity, it is not allowable to use cyclical structures in the string (which may be the basis of Jarek's exceptional solution). But she emphasises that Jarek's demonstration is based on 'good thinking'. This accolade for a wrong answer can be described as her

situational contribution to the "socio-mathematical norms" (Yackel and Cobb, 1996) for this classroom in this situation: New ideas are desired though they can fail.

Afterwards, the teacher holds up several quantities of pearls and each time one child restates the quantity first, followed by some addition terms fitting this quantity (mostly using the coloured partition). Thus, Jarek is the sponsor of several solutions and his first idea is the main idea of the emerging content, which develops in an increasingly experienced interaction structure. At the end of this period, the teacher holds up a string with 15 visible pearls and the teacher asks Jarek, again:

| 211 | Jarek | twenty-three minus eight |
| 212 | Teacher | very good *laughs* we haven't even calculated this far, very well done, I like that |

He creates a minus term, where the 15 pearls stand for the result of an abstract calculation; this is a new idea and he is the author of his response. With respect to aspects of arithmetic, the term is correct; however, it is out of the scope of the string (twenty pearls). It is assumed that Jarek already [in 112] knows about his experiment by answering seven minus zero. Jarek's answers demonstrate his affinity with presenting new ideas, testing the borders with a high risk of error. In terms of Huber and Roth (1999), Jarek is a typical "finding" pupil.

Marina's responsibility for the emergence of shared knowledge is quite different. In the already analysed part of the lesson, she was the first one who supported Jarek's idea. She is not responsible for the idea on her own, but at least for the acceptance and confirmation of this idea. Thus, her response contributes to the stability of the developing content which conforms to the usual classroom mathematics – so it reproduces a part of the culturally formed (mathematical) cognition. This stability depends on an approved solution pattern, and not on Jarek's innovation; also, he introduced it into the situation. So, Marina's achievement is the identification of the approved pattern in Jarek's responses.

This insistence on approved patterns is typical of Marina's participation and according to Huber and Roth (1999), she is a "seeking" student, which can be illustrated in more detail by the ongoing lesson: Each child receives their own string of pearls and the task reverses. The teacher is changing the demands:

256	Teacher	*forceful* fourteen fourteen show fourteen *pauses while the children are counting the pearls* look at which table the children *interrupting herself* please hold up so that we can see it
259		*All children hold up their string of pearls.*
260	Teacher	look yourselves, do you have the same findings at the table and compare

The teacher calls out a quantity and the children count this quantity 'on the string'. Then each child holds up their own string – so all the children present their solution to the teacher at the same time. But the teacher delivers the responsibility for correctness back to the children: They have to compare the strings at the table [260]. Let's have a look at Marina's table (the following transcript sequence reproduces

only the dispute between Marina and her neighbour Goran, so there are missing lines):

264		*Marina is looking at Goran's string of pearls. He has ten black and four pale pearls:* ●●●●●●●●●○○○○ *while Marina and most of the other children have chosen the inverted representation of fourteen* ●●●●○○○○○○○○○○
267	Marina	no, you are *forceful* wrong [lines 268 and 269 belong to the teacher and other children and did not concern the dispute between Marina and Goran]
270	Goran	*looking at his string* but *(inaudible)*
272	Marina	no, you must do it this way *showing her own representation of fourteen*

Marina's symbolising of fourteen with the string of pearls corresponds to the demonstrations of the teacher: Each time in the past sequences of this lesson, the teacher used the pale pearls for the ten and the dark pearls for the remaining ones. So Marina imitates the teacher's pattern. Goran modifies this pattern by reversing the colours. Marina rejects his solution. Unfortunately, Goran's reply is inaudible. Starting with *but* [270], it can be assumed that he defends his solution. This assumption is supported in Marina's repeated rejection, emphasising the colours again. Independently of the intentions of the teacher, Marina understands the actions of the teacher as an affirmation of the importance of the colours. She over-interprets the relevance of the colours. In doing so, she acts very conscientiously and this shows her general attention to the classroom discourse. This also shows her care in changing grasp patterns, too. Thus, her participation is more orientated towards supporting approved patterns. She is very concerned with holding up the 'right patterns'. This can be backed up by another response: The teachers asks, "Why is it possible to change the summands in an addition?", and Marina answers, *"Because you told us last week!".* This confirms the findings of Huber (2001) that certainty-oriented students depend on authorities.

CONCLUSION

People do not have the ability to convey meaning directly to other people; for example, teacher to pupils or pupils among themselves. Instead, each person endows objects (words, mathematical symbols, signs ...) with individual meanings, and these individual meanings are negotiated as taken as shared meanings and shared cognition in interaction processes – but there is now a possibility for a direct adjustment of different individual meanings. Thus, each interaction process is tainted with a high risk of misunderstanding. To reduce this risk, everyday interaction (like classroom interaction) is affected by routines and interaction patterns as well as by re-enacting the content. But every conversation is dependent on an adequate degree of new input; otherwise it is senseless re-enactment and will be broken down (if it is not a game). So there must be some new ideas or sufficent modifications – and this is evident in learning situations, too. The analysis has demonstrated how students jointly build on each other's ideas and how this process leads to establishing shared meanings: "Seeking" participation profiles (like Marina's) contribute to stabilising the

negotiating process, while pupils with a "finding" participation profile (like Jarek's) encourage the vitality of negotiating processes. Considering "learning-as-participation", the different participation profiles of individual learners jointly form the specific "participation room" of the classroom with its specific learning opportunities for all embedded learners. The teacher is only one participant among others who decides on the balance between innovation (by "finding" pupils) and stabilisation (by "seeking pupils") in this "participation room" but s/he has to exploit the different opportunities offered by the learners' profiles.

REFERENCES

Blumer, H.: 1969, *Symbolic Interactionism - Perspective and Method,* Prentice-Hall, Englewood Cliffs, NJ.

Brandt, B.: 2002, 'Classroom Interaction as Multi-Party-Interaction - Methodological Aspects of Argumentation'. In: J. Novotná (eds.): *Proceedings of CERME 2*, pp. 377-385. (http://ermeweb.free.fr/doc/CERME2_proceedings.pdf, retrieved March, 19, 2007)

Brandt, B.: 2004, *Kinder als Lernende,* Peter Lang, Frankfurt am Main.

Brandt, B.: 2006, 'Kinder als Lernende im Mathematikunterricht der Grundschule', in H. Jungwirth and G. Krummheuer (ed.), *Der Blick nach Innen*, Waxmann, Münster et al., pp. 19-52.

Bruner, J.: 1983, *Child's talk,* Norton, New York et al.

Bruner, J.: 1990, *Acts of meaning,* Harvard Univesity Press, Cambridge.

Erickson, F.: 1982, 'Classroom discourse as Improvisation. Relationship between Academics Task Structure and Social Participation Structure', in L.C. Wilkinson (ed.), *Communicating in the classroom,* Academic Press, New York, pp 153-181.

Goffman, E.: 1974, *Frame analysis,* Harvard University Press, Cambridge.

Goffman, E.: 1981, 'Footing', in E. Goffman (ed.), *Forms of Talk,* Basil Blackwell, Oxford, pp. 124-150.

Huber, G.L. and Roth, J.W.H.: 1999, *Finden oder suchen?* Huber, Schwangau.

Huber, G.L.: 2001, 'Kooperatives Lernen im Kontext der Lehr-Lernformen', in C. Finkbeiner and G. Schnaitmann (ed.), *Lehren und Lernen im Kontext empirischer Forschung und Fachdidaktik*, Auer, Donauwörth, pp. 222-245.

Knipping, C.: 2004, 'Argumentation structures in classroom proving situations', M. Mariotti, *Proceedings of CERME 3,* (http://fibonacci.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG4/TG4_list.html, retrieved March, 19, 2007).

Kovalainen, M. and Kumpulainen, K.: 2005, 'The Discursive Practice of Participation in an Elementary Classroom Community', *Instrutional Science*, 33, pp. 213-250.

Krummheuer, G. and Brandt, B.: 2001, *Paraphrase und Traduktion,* Beltz, Weinheim.

Krummheuer, G.: 2007, 'Argumentation and participation in the primary mathematics classroom.' *Journal of Mathematical Behaviour,* (in print)

Krummheuer, G.: 1995, 'The Ethnography of Argumentation', in P. Cobb and H. Bauersfeld (ed.), *The Emergence of Mathematical Meaning: Interaction in Classroom Cultures,* pp. 229-270.

Levinson, S.C.: 1988, 'Putting Linguistic on Proper Footing. Explorations in Goffmans Concepts of Participation', in P. Drew and A. Wootton (ed.), *Erving Goffman - Exploring the Interaction Order,* Polity Press, Cambridge, pp. 161-227.

Mead, G.H.: 1934, *Mind, self and society,* University of Chicago Press, Chicago.

Naujok, N.: 2000, *Schülerkooperation im Rahmen von Wochenplanunterricht,* Dt. Studien-Verlag, Weinheim.

Sutter, T.: 1994, 'Entwicklung durch Handeln in Sinnstrukturen. Die sozial-kognitive Entwicklung aus der Perspektive eines interaktionistischen Konstruktivismus', in T. Sutter and M. Charlton (ed.), *Soziale Kognition und Sinnstruktur,* Bis, Bibliotheks- und Informationssystem der Universität Oldenburg, Oldenburg, pp. 23-112.

Toulmin, S.E.: 1958, *The uses of argument,* Cambridge University Press, Cambridge.

Voigt, J.: 1984, *Interaktionsmuster und Routinen im Mathematikunterricht,* Beltz, Weinheim, Basel.

Voigt, J.: 1995, 'Thematics Pattern of Interaction an Sociomathmetical Norms', in P. Cobb and H. Bauersfeld (ed.), *The Emergence of Mathematical Meaning: Interaction in Classroom Cultures,* Lawrence Erlbaum, Hilldale, NJ, pp. 163-228.

Yackel, E. and Cobb, P.: 1996, 'Sociomathematical Norms, Argumentation, and Autonomy in Mathematics', *Journal for Research in Mathematics Education*, 27(4), 458-477.

---

[i] The German transcripts include prosodic aspects of the utterances, which cannot be transferred adequately into another language (e.g. raising pitch at the end of an utterance marks a question in German language). In this paper, information from prosody are added to the comments in italic – being aware that this is an interpretation (like transcribing in general).

MODELLING CLASSROOM DISCUSSIONS AND CATEGORIZING DISCURSIVE AND METACOGNITIVE ACTIVITIES

<u>Elmar Cohors-Fresenborg</u> & Christa Kaune

Institut für Kognitive Mathematik, Universität Osnabrück, D-49069 Osnabrück

In the last fifteen years of international discussions about mathematics education, there has been an increasing drive to make metacognition a central component of mathematics teaching. In our paper, we first present the framework of αβγ–automata as a mathematical model to describe the interaction between external and mental representations in discussions. Then we present a system for categorizing metacognitive activities during stepwise controlled argumentation in mathematics lessons with the categories monitoring, reflection *and* discursivity. *Both theoretical tools will be used for the analysis of a discussion between students which deals with the problem of whether $0.\overline{9} = 1$ is true, and the interplay of external and internal mental representation of the things being said and those being meant.*

INTRODUCTION

In the last ten years of international discussions about how to improve learning mathematics, one focus has been on students' metacognitive activities. An overview of early approaches to research in mathematics education concerning metacognition can be found in Schoenfeld (1992). In the field of educational psychology, Boekaerts emphasised (1996, 1999) which role metacognition plays regarding self-regulated learning. In our research project „Analysis of teaching situations for the training of reflection and metacognition in mathematics teaching in forms 7 to 10 at grammar schools" [1] we have analysed, in detail, the mechanisms which promote students' metacognitive activities by means of video documented teaching examples.

Often, the analysis of the components of metacognition is based on situations in which a *mathematical problem is to be solved* (e.g. Schoenfeld, 1992; De Corte et al., 2000; Kramarski & Mevarech, 2003). Therefore one important component is the *planning* of problem solving steps with suitable mathematical tools. On the other hand, the use of the tools has to be controlled, an analysis of the latest state of what has been achieved is necessary; a comparison with the goals set has to be made. The administration of this controlling and comparison is called *monitoring*. A third component is *reflection* on the given problem as well as on the understanding of concepts.

The focus of our research is on classroom discussions, in which the understanding of concepts, the use of algebraic tools, the invention of definitions and proofs and their understanding plays an important role. One objective of metacognition is to judge the adequacy of the production of representations from ideas or to carry out the steps backwards from representation to the presumed ideas of classmates. *Discursivity* is a characteristic of such discussions. Deeper understanding is only possible if the

monitoring and the reflection are precise. Therefore, discursivity in the discussions is needed for a classroom culture which promotes students' metacognitive activities.

In our paper we first present the framework of αβγ–automata as a mathematical model to describe the interaction between external and mental representations in discussions. Then we present a *system for categorizing metacognitive activities during stepwise controlled argumentation in mathematics lessons (CMDA)* with the categories *planning*, *monitoring*, *reflection* and *discursivity* (Cohors-Fresenborg & Kaune, 2005). Both theoretical tools will be used for the analysis of a discussion between students (14/15 years old) in grade 9, which deals with the problem, whether $0.\bar{9} = 1$ is true. [2]. Another good example for the importance of discursivity can be found in Boero & Consogno (this volume), which is exemplified in the additional paper "*Analysing a classroom discussion: alternative approaches*" (this volume).

MODELLING WITH αβγ–AUTOMATA PROCESSES OF UNDER-STANDING

As an important component in modelling discussions we want to take in single components the process of how persons imagine other people's ideas by understanding their talking or writing (external representations). As a metaphor (according to Lakoff, 1980), we choose the theory of αβγ–automata, by means of which Rödding (1977) modelled mechanisms of social behaviour: An individual, modelled by an αβγ-automaton, receives in state s, in which the person's knowledge has been coded with the help of the function α, a piece of information i, which also includes the situation, from its environment. The α-transition produces an idea, an inner representation; the β-transition describes inner mental processes in the individual. By means of a γ-transition, the individual carries out an action, e.g. passes on a piece of information to the outside, produces a (written or oral) description, an outward representation of its idea.

We are now going to look at a network of two αβγ-automata:

An external representation i is given, which serves as an input for person 1 as well as for person 2. Person 1 in state s_1 forms an idea of information i with the help of α_1, which leads to a new state, and processes it with β_1. With the help of γ_1, the person passes a representation of $\beta_1(\alpha_1(s_1, i))$ on to the outside. Person 2 perceives this representation in state s_2 and forms an idea of it by means of α_2. Moreover person 2 has – like person 1 – realized the external representation i and has formed an idea $\alpha_2(s_2', i)$ of it with different knowledge s_2'. He/she compares it with $\alpha_2(s_2, \gamma_1(\beta_1(\alpha_1(s_1, i))))$. Let us presume that these two do not fit together and that person 2 assumes that the reasons for that do not lie in his/her own thinking processes α_2. Person 2 can now suppose that person 1 has made a mistake in the representation (mistake regarding γ_1) or a mistake in his/her logic when processing $\alpha_1(s_1, i)$ by β_1 or that he/she has formed a misinterpretation of i (mistake regarding $\alpha_1(s_1, i)$). The thinking about which of the cases mentioned above is plausible belongs to the field of metacognition.
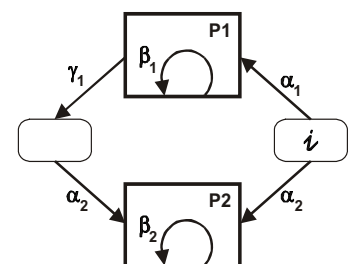
The process mentioned above describing the analysis of ideas and their representations will be shown with a hypothetical example from naïve set theory: What does it mean that two sets A and B have to be considered "together". Person 1 gives the formal representation A∩B. Person 2 can, on one hand, suppose that a representation or writing error (γ_1) has occurred or, on the other hand, that person 1 has got a false idea (a combination of β_1 and α_1) of how the word "together" has to be expressed. Person 2 supposes in both cases that his/her own mental constructions $\alpha_2(s_2', i)$ and $\alpha_2(s_2, \gamma_1(\beta_1(\alpha_1(s_1, i))))$ have worked correctly. By means of a question from person 2 addressed to person 1, person 2 can exclude that a representation mistake has been made. Person 2 presumes by means of his/her own knowledge that person 1 has become a victim of the misconception that the mutual contemplation of the sets A and B in the term, which has to be constructed, have to be expressed by the logical composition "and" (instead of "or").

CATEGORIZING METACOGNITIVE AND DISCURSIVE ACTIVITIES

The development of our *system for categorizing metacognitive activities during stepwise controlled argumentation mathematics lessons (CMDA)* started with analysing discussions in mathematics lessons concerning school algebra. This means activities concerning mathematical notations, term rewriting and solving equations. All these have in common, that single steps have to be justified by rules (theorems, definitions). The classroom discussions deal, for example, with the correctness of transformations or the justification of symbolic notations, the analysis of errors or misconceptions. But also, the question to what extent the things said (written) express the things meant can be a matter of discussion. Soon it became obvious that an extended version of *CMDA,* with more abstract formulations of the categories, is also useful to analyse other mathematical discussions in which argumentations are based on definite statements and controlled stepwise.

CMDA consists of the categories *planning*, *monitoring*, *reflection* and *discursivity*. Each of these consists of several subcategories, which have different aspects. For each of them, it can be judged whether the activity is done by a teacher or a student. By means of these decisions, metacognitive or discursive activities of teachers and students can be categorized by one system. You will find on the next page the system without the column for *planning*.

When analyzing a transcript we use a specific code for the (sub)categories and their aspects. This code consists of (up to) 4 characters, possibly with an additional prefix.

		Monitoring			**Reflection**			**Discursivity**
1	M1	Controlling of **Calculation**	R1	Reflection on **Concepts**		D1	Initiating of a discourse	
	M1a	locally (finding possible mistakes)	R1a	assignment of an object / an issue to a concept		D1a	invitation to a „limited" discourse (e.g. question concerning agreement/ assurance)	
	M1b	successive (complete) checking	R1b	classification of a concept into a concept hierarchy		D1b	invitation to an (initiating of) an extended discourse	
	M1c	checking of a fictitious calculation step				D1c	... by revealing a discrepancy between something said and meant	
						D1d	... in order to avoid a false idea / a deficiency of understanding	
						D1e	..., in order to generally present facts	
						D1f	... in order to check a plan, a planning step/ a strategy	
2	M2	Controlling of an assessment of (assumed) **Mistakes** (when calculating or rewriting)	R2	**Consciousness of Method** Comment on a (possibly wrong) application / employment of mathematical tools		D2	**Linking** of a discourse contribution	
	M2a	naming	R2a	having noticed the effect of a general tool in a specific case		D2a	naming of reference points or persons	
	M2b	uncovering one's own false ideas or those of others	R2b	linked use of tools		D2b	deliberate (precise) reference to a former contribution, not to the directly preceding one	
	M2c	uncovering one's own fundamental or common false ideas or those of others	R2c	having realized that after the use of a tool a better starting position for further steps has been reached		D2c	securing of the conversation basis / linking of the following or preceding statement	
						D2d	repetition of the things said as a basis for further reasoning	
						D2e	assurance of things said/ meant	
						D2f	prompt setting off of a contribution against another (partial) contribution	
3	M3	Controlling of **Terminology** and **Notation**	R3	**Analysis of Structure** of a term / mathematical expression		D3	Discursitivity **with oneself**	
			R3a	without taking into consideration any rewritings				
			R3b	rewritings taken into consideration				
4	M4	Controlling of **Argumentation**	R4	**Analysis of Structure of** reasoning / **discourse**		D4	**Education** for discourse	
	M4a	local examination	R4a	implementation		D4a	Revealing / agreement on rules for the discourse	
	M4b	stepwise / global examination	R4b	assessment of a wrong reference or contradiction		D4b	Adherence to the rules of the discourse	
	M4c	uncovering mistakes in the argumentation	R4c	assessment of change in point of view, choice of word, concept, way of reasoning				
5	M5	Determination of position: naming of **Deficiency of Understanding**	R5	Willful choice of a **Representation** (formula, graphic, term, text passage, etc.)		D5	Negative discursivity	
	M5a	defined step	R5a	identification / marking		D5a	repetition of things already said without adding a new point of view	
	M5b	globally	R5b	special (term)-representation		D5b	introduction of alternative statements or proposals without setting them off against others	
			R5c	like a or b, in order to specifically encourage understanding		D5c	statements/ questions are not recognizable referring to the things occurred or said	
			R5d	like a or b, in order to stimulate or support abstraction processes		D5d	Non-commented change of reference point / meaning of a word	
			R5e	like a or b, in order to stimulate metacognition		D5e	inadequate choice of words for description or comments	
						D5f	No intervention taken against severe disregard of discursivity rules	
						D5g	asking a leading question	
6	M6	Determination of position: naming of **Planning Deficiency**	R6	**Reflecting Assessment / Evaluation**				
	M6a	next step or steps respectively	R6a	(one's own) proceeding, drawing an (interim) balance				
	M6b	globally	R6b	(one's own) strong or weak points				
			R6c	important points, ideas, general difficulties				
7	M7	Monitoring the **Reference to facts and aims**	R7	Picking out the interaction of **Representation** and **(false) Idea** as a central theme				
8	M8	**Self-Monitoring**						
	M8a	own calculation						
	M8b	expressions or terminology						
	M8c	reasoning						

The first character is the first letter of the category's name; the second character is either "S" for student or "T" for teacher; the third character is the number of the subcategory; the fourth character is a small letter indicating the aspect of a subcategory.

If a reason or an explanation is given for one activity, then the letter r is set as a prefix to the code.

If it has to be indicated for one subcategory (or its aspects) that the specific activity is demanded then the letter d is set as a prefix to the code.

Colours are affiliated to the categories *planning*, *monitoring*, *reflection* and *discursivity* (except negative discursivity) and their codes.

In the transcript, the code is set in the right-hand column at the level of the appropriate piece of text. The classified pieces of text are coloured, too. This supports a more general perspective when analyzing the transcripts. In a transcript, it may happen that single words or parts of a sentence belong to another category than the surrounding text. This leads, of course, to a different colour. In the special case that the classified subcategory or aspect belongs to the same category, this change cannot be made visible by colours. Therefore this part (and the code) is typed in **bold** letters. The pieces, where a teacher is talking, are additionally marked by under-lining. The classification of an utterance is done only according to the format. It is not considered whether the claims (for example "… there is a mistake …") are true.

For a more quantitative analysis of a transcript the given codes as well as the attached line-numbers can be transferred into a file. It enables the computing of profiles concerning metacognitive and discursive activities, both for a lesson and a teacher. For details concerning *CMDA* see Cohors-Fresenborg & Kaune (2005).

ANALYSIS OF A TRANSCRIPT

In the following, we will use the two analyzing tools which we have presented. We have chosen a transcript of a lesson (whole class teacher led) which deals with the acceptance of the validity of the equation $0.\bar{9} = 1$, or in words, that both terms are names for the same figure. This keeps causing problems to pupils of all age groups (see, for example, Tall, 1977). The extract from a transcript (on the following page) shows the struggle of a group of pupils to bring their ideas into line with the representation.

Analysis with the framework of αβγ−automata

The starting point of the transcribed discussion mentioned above is Jens' statement that there is no figure between $0.\bar{9}$ and 1. In our analysis, this is taken as input i. Mona is person 1, who picks up this statement ($\alpha_1(s_1, i)$) with the help of her pre-knowledge s_1, and reflects. As a consequence *"the figure that you would need in order to make zero point periodic continued nine a one"* comes to her mind (she only explains this in lines 14/15). These ideas are described by $\beta_1(\alpha_1(s_1, i))$. She then says: *"It may, however, be zero point infinite zero and then a one."* (γ_1 in lines 5/6).

Jens:	... but Peter says that zero point nine is the same as one and that otherwise there always	DS2d
2	has to be a figure in-between ... that there has to be at least one figure between two	MS8b
	decimal numbers. **And in this case, it isn't** and therefore it is logical that this should	**MS4a**
4	actually be correct.	MS4b
Mona:	Well, I do think that there is a figure. It may, however, be zero point infinite zero and	DS2f
6	then a one, some time or other	DS5d
	[*she laughs*] ...	MS8b
8 T.:	Could you please write it on the board, how you imagine this (figure)?	*d*RT5e
Mona:	No, not really, ... as a figure it doesn't exist ... you can't write it down in this way. But	MS4c
10	logically it would be possible.	*r*RS7
	[*The following 38 sec. (Mona's slip of tongue) have been deleted.*]	
12 T.:	I would like to know if it is at all clear to everybody what Mona wanted to say, about	*d*DT2c
	what figure she has been talking. **She said: "I cannot write it down."**	**DT2d**
14 Mona:	Well, I meant, the figure that you would need in order to make zero point periodic	DS2c
	continued nine a one. That's the figure I have been talking about. If first there are many,	
16	many zeroes and then at some time or other a one, but this doesn't exist in principle.	MS8c
Suse:	This is what I also wanted to say: There is an infinite number of nines behind the zero,	DS2d
18	hm, periodic continued nine, ... and she thinks that, that there should be a figure which	MS8b
	has exactly as many zeroes, which means infinite zeroes, ... well there is a one at the	DS2d
20	end so that, if you add it, you obtain one. That's the figure she is looking for. But you	*r*MS4c
	cannot write it down because there would have to be an infinite number of zeroes.	
22 T.:	Yes, let's pick up some more ideas. You said, we have got two positions. (...)	DT5c
	[*No reaction of pupils during the next 14 seconds.*]	
24 Suse:	I would say Peter's solution is correct, because, if you take different figures instead, if	*r*MS4b
	you take for example five and two instead of zero point periodic continued nine and	
26	one, you know that they are not the same, as three is between them. And regarding zero	
	point period continued nine and one, there is no figure between them. You know you	
28	cannot write down a figure. Mona, however, thinks that there should be this "zero point	DS2b
	period continued zero one"-figure, but you cannot write it down.	DS5d
30	Thus, it doesn't really exist a figure. And therefore this could be right.	*r*MS4c
	Mona.	DS1a
32 Mona:	Well, I only meant: the figure doesn't exist, but logically you could imagine it so	DS2d
	[*laughter*] That it could exist. Therefore ... [*murmuring*] .... **but this figure doesn't**	RS7
34	**exist. Hm, that is clear. It doesn't work.**	**RS6a**
T.:	Say it loud, please, Judith and Juli?	
36 Juli:	Yes, Judith and I are just trying to imagine the figure "zero point periodic continued	RS7
	zero one", but this is somehow weird.	DS2b
38 Jens:	I think that there cannot be a further figure behind a periodically continued figure.	MS2b
Suse:	Yes, that is right, yes, that is true, because the zero, **hm, because the periodic line is**	*r*MS2b
40	**above it,** which means, it is the zero that always repeats. Thus there cannot suddenly	**MS8c**
	be a one behind it.	DS2c
42	Which means this figure doesn't exist.	RS6a
	If at all, the periodic line would have to be above both figures, and then it would	**RS3a**
44	**be continued in that way: zero point zero one zero one zero one.** This would not be	RS6a
	the figure Mona meant.	DS2c

The sequence of Mona's remarks and her word choice suggest that the given figure is supposed to be the figure, which should exist. This may, however, not be the case.

Her first remark (a proof would be the first laughter in line 7) or the teacher's request (line 8) causes Mona (here we take her also as person 2, but in state s_2'), to once more sort out in her mind what she said ($\beta_2(\alpha_2(s_2', \gamma_1(\beta_1(\alpha_1(s_1, i)))))$). Her laughter in line 7 and her comments (lines 9/10) respectively are interpreted as an expression of having found a mistake in her long process of thinking. β_2 describes a metacognitive thinking process referring to her own cognition. In the formula, this is expressed by β_1.

The teacher (lines 12/13) causes Mona to repeat her remarks once more. In this new situation (represented by the state s_1'), Mona replies with a different formulation ($\gamma_1(s_1')$) *" ... If first there are many, many zeroes and then at some time or other a one ..."* (line 15/16). The process of the discussion, however, shows that she has changed her formulation, but not her opinion. We therefore take this change as a mistake in representation, ("*many, many*" is different to *"an infinite number"*) or as a variation in the representation ("many, many" with the meaning of "unlimited", i.e. "an infinite number"). The reactions of her classmates (Suse, Juli) show that they also take Mona's statement as a variation in representation.

For further analysis, Suse is person 3. In lines 28/29, she again refers to what Mona said first of all ($\gamma_1(\beta_1(\alpha_1(s_1, i)))$) (lines 5/6). This forms, together with the pre-knowledge, state s_3, in which Suse notices ($\gamma_3(s_3)$ in lines 27 to 29) that this number does not really exist as you cannot put it down in writing, as there is no formal representation. If there is no formal representation of the things having been said, it cannot be of any meaning, i.e. you cannot talk about something existing.

Juli refers to what Mona has said and gives a formal representation "$0.\overline{0}1$" and tries to imagine the figure represented in that way. Jens gets into the discussion on this representation level and criticises that this way of representation is not allowed as there cannot be another figure after a periodic number. This means Jens picks up the form of representation, checks its syntactic correctness and finds a syntactic error. Suse picks up Jens' idea using the semantics of the representation: The periodic line means that there is an infinite number of zeroes, and there cannot suddenly be a one.

The complete dialogue repeatedly deals with representations and ideas, with the assumptions of classmates, and which ideas other classmates might have (in the case of Mona it is even herself) when they have offered a representation. Then the classmates compare them with their own ideas. The pupils have a feeling for the fact that talking, as long as no gradual meaning can be related to the verbal constructions used, only *sounds* meaningful but does not really have a meaning.

The question to what extent verbal constructions can constitute meaning, plays a role when terms (as name replacements) are introduced by denomination operators. Mona introduces the figure that she means by a denomination term [3]. *"Well, I meant, hm, the figure that you would need in order to make zero point periodic continued nine a one"* (lines 14/15). Now the question arises if the use of a denomination operator is allowed. If the things said were actually the things meant, the figure would be unambiguously defined, i.e. it would be the figure zero. Everything would be in order and the use of the denomination term would be a name replacement for the figure

zero. That is, however, not meant, as mentioned above. Mona also talks about the past in lines 14/15 before she understands *"…but this doesn't exist in principle."* When she says in lines 32/33: *"…but logically you could imagine it so. That it could exist."*, she presumably means "… it could verbally be formulated in that way". As the figure does not exist, this verbal construction is not allowed *"Hm, that is clear. It doesn't work".* (line 34).

Analyzing the metacognitive and discursive activities

The first intervention of the teacher in line 8 has to be understood in such a way that she foresees that the expectation to put something down in a formal representation, where syntax and semantics are clearly defined, causes a gradual construction of meaning and mere talking becomes obvious. For such a type of intervention, we have constructed category *d*RT5e. In line 9 Mona controls the mistakes in her argumentation (MS4c). Then she detects a conflict between internal and external representation (*r*RS7). The teacher assumes that some students may have difficulties in following the argumentation. In lines 12/13 of her intervention, she ensures the basis of the conversation (*d*DT2c). She repeats Mona's sentence (line 13) as a basis for further reasoning (DT2d). In lines 14/15 Mona repeats her preceding statement (DS2c). Only this precise formulation enables Mona's following monitoring process: Up to now, she has only said that you can't write down this figure (MS4c); she formulates in line 16 that this figure doesn't exist. This is a monitoring of her own reasoning (MS8c).

For discursivity in classroom culture, it is necessary that the students themselves practise monitoring of their formulations, such as controlling of terminology and notation, because their classmates have to refer precisely to what has been said in their contributions (e.g. MS8 in lines 2/3, 7, 16, 18, 40/41).

From a content orientated point of view, we have to remark that the students' discussion deals with two "figures": On one hand, Jens (in lines 1-4) and Suse (in lines 24-28) both talk about a figure between $0.\bar{9}$ and 1; on the other hand, Mona (in lines 5/6, 9/10, 14-16) and Suse (in lines 18-21) both talk about a figure which describes the distance between $0.\bar{9}$ and 1. As Mona says in line 5 "*there is a figure*", although she talks about the distance, this utterance is marked as "negative discursivity" by DS5d (non commented change of meaning of a word).

The role of a teacher, who will promote discursivity in classroom culture, is to monitor the discourse concerning the difference between what is said (written) and what is meant, because he / she has to ensure that all students share the same conversation basis. In the case of discrepancy he / she has to intervene; otherwise there is only talking and not a goal-led discussion among the students or a lot of misunderstandings will arise. In this scene, the teacher makes four interventions: Two of them follow this demand (*d*RT5e in line 8, *d*DT2c in line 12 and DT2d in line 13). The third intervention (line 22) is marked as "negative discursivity" by DS5c, because her statement doesn't refer to the things said : there are not "*two positions*", but the students talk about different numbers. The forth intervention (line 35) is for classroom management only.

If there is an utterance in which, beside the categories "monitoring" or "reflection", an additional category "discursivity" is also applicable, we have introduced the possibility of double categorizing, which is marked in the text by green underline points (e. g. lines 36/37 and 45).

As an outcome of this scene, the students detect that no "figure" exists to describe the distance between $0.\bar{9}$ and 1. With this insight the tool "formal representation" which leads in a first step to "$0.\bar{0}1$" is important. This is rejected by a syntactical argument (line 38). In the following step the formal object "$0.\overline{01}$" is created (in line 43) and attached with a meaning afterwards (line 44). Then they detect (lines 44/45) that *"this would not be the figure Mona meant"*.

SUMMARY

In this paper we have shown that formalisation can be used as a tool to precisely analyse different aspects of language and communication in learning. The formalizations force one to decide precisely what is meant. By using the framework "network of αβγ−automata", the structure of discussions and the interplay between external and internal representation are detected. Additionally, by using the *CMDA*, the different γ−transitions and hypothetical α− or β−transitions can be categorized, if they are followed by a verbalisation or a gesture. All together, this methodology allows a deeper understanding of classroom discussions and gives hints to measure the teaching quality, used in the evaluation process of teaching and classroom culture.

NOTES

1. The project is supported by the Deutsche Forschungsgemeinschaft under reference Co96/5-1.

2. For further use of the theoretical framework of αβγ−automata for analyzing transcripts see Cohors-Fresenborg et al. (2001), and of the *CMDA* see Cohors-Fresenborg et al. (2005), and Kaune (2006). *CMDA* is developed and used for analyzing math lesson in naturalistic settings in grades 1-13.

3. The concept "denomination operator" has been introduced by Whitehead & Russel (1910, pp. 173-186) together with an analysis of its ambiguity. As the students have been taught according to the Osnabrück Curriculum (Cohors-Fresenborg, 2001) they are familiar with the thoughts about denomination operators ("definite article") (Cohors-Fresenborg, Griep, & Kaune, 2003, pp. 51-52). In the analyzed scene the question, What does "this figure" mean?, is essential for understanding.

REFERENCES

Boekaerts, M.: 1996, Teaching Students Self-Regulated Learning: A Major Success in Applied Research, in J. Georgas et al. (Eds.), *Contemporary Psychology in Europe*, Hogrefe & Huber, Seattle, pp. 245-259.

Boekaerts, M.: 1999, Self-Regulated Learning: where we are today, *International Journal of Educational Research, 31*, 445-457.

Cohors-Fresenborg, E.: 2001, Mathematik als Werkzeug zur Wissensrepräsentation: das Osnabrücker Curriculum, *Der Mathematikunterricht*, 1, 5-13.

Cohors-Fresenborg, E. and Kaune, C.: 2001, Mechanisms of the Taking Effect of Metacognition in Understanding Processes in Mathematics Teaching, in *Developments in Mathematics Education in German-speaking Countries, Selected Papers from the Annual Conference on Didactics of Mathem.*, Ludwigsburg 2001, SUB Göttingen, http://webdoc.sub.gwdg.de/ebook/e/gdm/2001/index.html, 29-38.

Cohors-Fresenborg, E., Griep, M., and Kaune, C.: 2003, *Sätze aus dem Wüstensand und ihre Interpretationen*, Forschungsinstitut für Mathematikdidaktik, Osnabrück.

Cohors-Fresenborg, E. and Kaune, C.: 2005, *Kategoriensystem für metakognitive Aktivitäten beim schrittweise kontrollierten Argumentieren im Mathematik-unterricht.* Arbeitsb. Nr. 44. Forschungsinstitut f. Mathematikdidaktik, Osnabrück.

De Corte, E., Verschaffel, L., and Op't Eynde, P.: 2000, Self-regulation: A characteristic and a goal of mathematics education, in M. Boekaerts, P. R. Pintrich, and M. Zeidner, *Handbook of self-regulation*, Academic Press, San Diego, pp. 687-726

Kaune, C.: 2006, Reflection and metacognition in mathematics education - tools for the improvement of teaching quality, *Zentralblatt f. Didaktik d. Mathem*, 350-360.

Kramarski, B. and Mevarech, Z. R.: 2003, Enhancing Mathematical Reasoning in the Classroom: The Effects of Cooperative Learning and Metacognitive Training, *American Educational Research Journal*, 40, 1, 281-310.

Lakoff, G.: 1980, *Metaphors we Live By*. The University of Chicago Press, Chicago.

Rödding, W.: 1977, On the Aggregation of Preferences, *Naval research Logistics quarterly*, 25/1, 55-79.

Schoenfeld, A. H.: 1992, Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics, in D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*, Macmillan, New York, pp. 334-370.

Tall, D.: 1977, Conflicts and Catastrophes in the Learning of Mathematics, *Math. Education for Teaching,* 2, 4, 3-18.

Whitehead, A. N. and Russell, B.: 1910, *Principia Mathematica*, Vol. I., Cambridge University Press, Cambridge.

THE LANGUAGE OF FRIENDSHIP: DEVELOPING SOCIOMATHEMATICAL NORMS IN THE SECONDARY SCHOOL CLASSROOM

Julie-Ann Edwards

School of Education, University of Southampton, UK

This paper reports on a study of friendship groups as they learned mathematics in small groups in a secondary school classroom. It examines the role that discussions between friends have on their ability to negotiate taken-as-shared meanings (or sociomathematical norms). Transcripts of peer talk in a low attaining group of 14-15 year olds are analysed for evidence of the sociomathematical norms which were found in a study by Cobb et al (1995) with 6-8 year olds. Findings suggest that similar negotiations are evident, despite the differences in age, but that an additional sociomathematical norm related to mathematical efficiency in written communication is identified. The focus of this paper is a socioconstructivist analysis of students' talk rather than a sociolinguistic analysis.

THE NATURE OF FRIENDSHIP

The study of friendship is undertaken in three fields of study – anthropology, psychology and sociology. Each offers its own perspective on the nature and function of friendships. Despite the multitude of studies, Allan (1996) notes that there is a lack of firmly agreed and socially acknowledged criteria for what makes a person a friend. From an anthropological perspective, Pahl (2000) offers a definition of friendship which fits the research setting described here:

> Friendship is a relationship built upon the whole person and aims at a psychological intimacy, which in this limited form makes it, in practice, a rare phenomenon, even though it may be more widely desired. It is a relationship based on freedom and is, at the same time, a guarantor of freedom. A society in which this kind of relationship is growing and flourishing is qualitatively different from a society based on the culturally reinforced norms of kinship and institutional roles and behaviour (pp163-4).

Bell and Coleman (1999) similarly argue an anthropological stance that a Western view of friendship is a matter of choice and that "friendship becomes a special relationship between two equal individuals involved in a uniquely constituted dyad" (p8). However, the research undertaken here is with friendship groups of between three and six individuals. Allan (1989) suggests that even in the dyadic context, friendships are a matter of opportunity, dependent on class, gender, age, ethnicity and geography. This is reflected in the discussions amongst friends in the research study.

In psychological studies, there is a linking of developmental stages in friendship with Piagetian stages of development. For example, in developing notions of empathy and the ability to see the point of view of another, Erwin (1993) outlines Selman's (1980) model of the stages of development in 'role-taking'. Note that this 'role-taking' is different from that related to work in groups. I outline the final two of five stages, as

these pertain best to the age of the students in the study in this paper. Selman's fourth stage is called 'Mutual role-taking' and occurs at approximately 10-12 years of age. This involves the child in being able to recognise the relationship of their own perspective to that of another and in appreciating that others are also aware of their perspective. The fifth stage, 'Social and conventional system role-taking', begins between 12 to 15 years and continues into adulthood. This is when general social considerations, rules and norms are taken into account and reflected upon. The complexity and subjectivity of other people are recognised, as are their consistent patterns of personality and behaviour. Given the age of the students in this wider study (11-15 years), it is expected that these two stages will be evident in the discussions.

Sociological studies examine the impact of friendships on individuals and in social contexts. Adams and Allan (1998) state that friendships cannot happen in a social or economic vacuum:

> Relationships have a broader basis than the dyad alone; they develop anad endure within a wider complex of interacting influences which help to give each relationship its shape and structure. If we are to understand fully the nature of friendships, these relationships need to be interpreted from a perspective which recognises the impact of this wider complex (pp2-3).

Gottman and Parker (1986) describe the particular social skills which are developed within friendships. The final six of these are:

> conform, cooperate and compete
>
> take risks
>
> develop communication skills
>
> develop negotiation skills and tact
>
> resolve conflicts
>
> develop shared meanings for group interaction (p282)

These six skills are particularly relevant to the study of friendships in mathematics classrooms. In other studies of young children working in friendship groups, these skills are similarly identified. Schneider (2000) reports Nelson and Aboud's (1985) study which found that friends explained their opinions and criticised their partners more often than non-friends. They argued that "higher levels of disagreement led to more cognitive change than did compliance" and concluded that "friends who experience conflict undergo more social development than non-friends do in conflict" (p76). The reasons given for this were that friends were likely to alter their opinion in favour of the more mature solution, whereas in non-friend pairs, either in the pair was likely to change their opinion. This has implications for friends working in groups in mathematics classrooms, as there may be a parallel in friends opting for the more mathematically different, mathematically sophisticated, mathematically efficient or mathematically elegant solution, whereas non-friends may not do so as readily.

RESEARCH ON SOCIOMATHEMATICAL NORMS

In order to explain the nature and development of sociomathematical norms in classrooms, I intend to focus on the work of six researchers (three American and three German), undertaking research from psychological and sociological perspectives on the same data collected over a period of 10 weeks in second and third-grade (6-8 year olds) US classrooms during a year-long classroom experiment in inquiry-based classrooms. Bauersfeld, Cobb, Krummheuer, Voigt, Wood and Yackel, define the study as a 'teaching experiment classroom'. Lessons typically consisted of a teacher-led introduction to a problem as a whole class activity, cooperative small-group work in pairs, and follow-up whole class discussion where children explain and justify solutions to each other. Recordings were taken of all small-group sessions and whole-class discussions on an arithmetic topic and these tapes were analysed. Small-group interactions were analysed on the basis of their "taken-as-shared" mathematical meanings that were established within the group (Cobb 1995). The teacher actively guided this establishing process. Cobb describes small-group norms as including:

> explaining one's mathematical thinking to the partner, listening to and attempting to make sense of the partner's explanations, challenging explanations that do not seem reasonable, justifying interpretations and solutions in response to challenges, and agreeing on an answer and, ideally, a solution method (p 104)

Interactions between children were identified as univocal explanation (in which one child assumed the authoritative position) or multivocal explanation (in which explanations and solutions were joint). A definition of authority was only accepted if the non-authoritative child accepted the authority of the other. Some children found multivocal explanations difficult because they had not established a 'taken-as-shared' basis for their discussion. However, only multivocal explanation was considered productive in its outcome. Direct collaboration, in which roles were assigned to meet the desired outcome, was deemed non-productive. Indirect collaboration, in which children appeared to be working independently whilst talking aloud, was considered productive because children found what each other were saying significant for them at the time.

These six authors assert that in a mathematical environment, the social norms that are interactively established in groups in any setting take on particular features specific to mathematics. These were recognised from tape recordings of lessons by identifying regularities in the patterns of social interactions. The authors argue that, whilst children should be challenging each other's thinking and justifying their own thinking in any area of the curriculum, in mathematics there are particular norms set up within groups as to what is taken-as-shared meaning about acceptable mathematical explanation and justification.

The premise upon which sociomathematical norms are established is that children understand that the basis for explanation is mathematical rather than status-based

(e.g. explaining for authority). Yackel and Cobb (1996) argue that these norms are established in stages of development. The first is explaining as a description of procedure, i.e. instructing how to do an act; the second is explaining as describing actions on a real (mathematical) object; the third is accepting this second stage as an object of reflection and deciding if it is valid for others. These can be interpreted as stages of computation, conceptual explanation and reflective action.

This exploration of a sociomathematical norm as determining an acceptable mathematical explanation serves to illustrate other sociomathematical norms identified. These include what counts as mathematically different, mathematically sophisticated, mathematically efficient and mathematically elegant. In negotiating sociomathematical norms, children become increasingly autonomous, the authors argue. They provide evidence of increased learning opportunities through listening and challenging the explanations of others.

I argue that friendship groups in mathematics classrooms of 11-15 year olds, in particular, offer the opportunities for these sociomathematical norms to be negotiated effectively. The following, from a study of friendship groups, offers evidence for the stages of developing sociomathematical norms and suggests differences because of the relative ages of student participants in the study.

THE STUDY OF FRIENDSHIP GROUPS

Students in this study (Edwards, 2003) attended an inner-city comprehensive secondary girls' school of 1087 students in the south of England. This population represented a full social and ethnic mix, with the majority of girls of white background, though there is a significant minority of 22% Asian girls and a total ethnic minority of 28%. The department operated a problem-solving curriculum based on the activities of the Graded Assessment in Mathematics (GAIM) project. These activities were introduced as a whole-class discussion, with students and teacher making possible suggestions for routes for exploration. Most of the subsequent work was in small groups of two to six students, though the class was sometimes drawn together at various points to enable a student to explain a discovery or the teacher to make a teaching point from something that has arisen. The teacher circulated amongst the small groups, supporting thinking, and assisting the direction of the activity. Small-group organisation was on a self-selected friendship basis but some groups were reorganised or split if they become mathematically unproductive.

Audio-recordings of whole-class and small-group interactions were taken over a period of eight weeks for a high attaining Year 9 group (13-14 year olds) for all lessons covering two GAIM activities. A low attaining Year 10 class (14-15 year olds) was recorded for some of its lessons over a period of two weeks using the same GAIM activity undertaken by a middle attaining Year 7 class (11-12 year olds) and this Year 7 class was recorded over the same period of time. The recordings were

taken in the third term of schooling when these groups had been working together for approximately 24 school weeks. Evidence from a Year 10 group is presented here.

EVIDENCE FROM WORK IN FRIENDSHIP GROUPS

The full transcript of the lesson for F, R and Z (Year 10) from which this example is taken gives strong evidence for the three levels of establishing sociomathematical norms for mathematical explanation. Their levels of questioning and understanding develop from procedural through conceptual to bordering on reflective. F, R and Z are completing an activity in which they are agreeing a solution for finding the number of possible half time scores for a Hockey match, given any final score. Initially, they focus on procedure:

> Z Now two times three ..
>
> R two times one is two
>
> Z Yeah
>
> F Add two......
>
> R add four
>
> Z Yeah, both those ... equals six …two add two .. two times two is four, is it ..? Yeah Add that, add that is nine … two times three is six .. Oh, maybe not
>
> R Yeah but that's not ... that's the unacceptable one, innit.
>
> Z I'll just see this one
>
> R I'll ask her. Miss? (T arrives)
>
> T two times one is two
>
> Z Is it that, Miss, look … two times one is two, add two, add two, equals six … two times two is four add that add that is nine …What am I doing about ..? two times three is six Oh that doesn't work But it does work over here …three times one is three add that add that equals eight
>
> T Does it work for this one?

Later, after an intervention from the teacher, they then focus on the reasons why they need to have the solution they have derived. This demonstrates the conceptual level described by Yackel and Cobb (*ibid*).

> T Think about *why* you need to add one each time ... What have you got there?
>
> Z Four sets of group, um, four sets of goals, ohh
>
> R I know Miss
>
> Z What is it?
>
> R We can add one to 0 to get our next .. things and then one, to .. you add another one to one to get two
>
> F Yeah but *why*? The reason *why*, not what you do

R Yeah, *why*, right What she's asking us this bit, yeah, *why* do we need to add one to that. The reason is that we need to add one to that first to get that

F Which means, in many more words, is you need to add one to get your answer.

R No

Z But she said *why* didn't she

R That ain't the answer. That ain't the answer. That's not answer.

F No, but the answer to why .. is why you have to add one to get the answer

R Is because you add one

The final stage of this development of sociomathematical norms is demonstrated clearly when these girls are considering the impact of the written communication of their solution. They are writing the reason why they needed to add one to each number in their solution: Half time scores = (n+1) x (y+1). They are attempting to write their verbal description of needing to add one each time because they are including zero in the total. Although they are not at the stage of fully reflecting on this communication, they are at the stage of recognising its impact and importance. They are using their explanation as an object for a focus for activity.

F The .. reason .. why .. you .. add .. one ...   [as she writes]

R To what .. what do we add one to?

F Add ... one ...

Z To ..

F Add one, right, to each goal

R To each set

F Yeah, to each goal number

R To .. each .. goal .. number ...   [writing]

Z Is .. because ..

F Because ..

Z If it was ..

F Because .. hang on .. because we started off with zero

R We always included zero

F Because .. we .. start .. off .. with .. zero .. and we have to add one all the time. That's it

Z Because we start off with zero and what?

F We have to .. add

R Zero. Have you got ...

Z What?

F Add ... include zero

R And we need to add one

F And we have to add one

R We have to move .. to goals

Z To make it up to another number, add one

R No, later on .. Cos it's the next ..

F We have to add on zero, start with the zero because

At this point they are confused about the difference between adding on zero and starting with zero. However, the continued extract shows that this confusion is only a function of the writing as, between the three students, they sort out an acceptable written explanation.

Z Start off with zero

F Because ..

Z Zero, and to add on another number

R One, add one because ..

Z Read that

F Add one to the set of goals, hang on, we .. add .. one .. to .. the .. set .. of .. goals .. because ..

R We need to move onto the next one ...

F Because we started ..

R We need to go onto the next number

F With ..

Z Yeah

F Zero Zero [reading] We add one to the set of goals because we started with zero and ..

Z We need to go onto the next number

F We .. need .. to .. move .. on .. to ..the .. next .. number .. which .. is ..what .. we .. started .. with. What do you think of that?

These low attaining Year 10 girls who are working towards their algebraic solution: Half time scores = (n+1) x (y+1) know that they are refining their mathematical efficiency through symbolism. This provides an example of a different sociomathematical norm being established than those identified by Yackel and Cobb and is similarly identified in the Year 7 and Year 9 groups. Although this is not an example suggested by Yackel and Cobb, I believe that it is, equally, an example of a sociomathematical norm at this age level because the students establish a taken-as-shared meaning for this important element of communicating mathematics. The reason it may not be identified in Yackel and Cobb's work is because their research

was done with elementary school children where written recording of work may not be a focus of activity.

Talking aloud is a significant and prevalent feature of all the groups studied. Noddings (1990) suggests that the level of elaboration required by talking aloud forced the student to concentrate on the problem. In the extract above, the teacher was not present during most of the time these students talked aloud as they wrote their solutions. However, the extent to which the purpose of talking aloud, in this case, is to keep them focused on the problem is debatable. I believe the purpose is related more to refining their own constructions of the solution.

The development of explanation and justification is an essential component of group work if students are to benefit from the trust established in friendships. In all the recordings there is a drive by group participants to generate a solution that they knew would work. Much of this knowing comes from questioning each other, arguing and justifying decisions to each other. Throughout the recordings there is also evidence of enjoyment in the form of laughter about mathematical situations that arise and a gentle banter about own performance or ability or that of another's. Rodgers (1995) argues in support of this enjoyment when she says "All the evidence points to the fact that the use of humour and laughter are very useful in dissipating the tensions created by learning difficulties" (p 36). The familiarity of friends in the context of mathematics groupings is a mechanism by which tensions relating to mathematics are more easily addressed (Edwards, 2004).

DISCUSSION

The sociomathematical norms identified in this study are almost all based on mathematical explanation, as are those of Cobb *et al* in their study. Norms of mathematical difference, mathematical sophistication, and mathematical elegance are not identified, though examples of mathematical efficiency in communicating are identified in the older age groups.

The difference in age groups in this study and that of Yackel and Cobb raise issues of comparability. The level of mathematical language used in secondary classrooms is already more sophisticated than that in elementary classrooms. This makes analysis of small group talk to determine whether the group is establishing taken-as-shared meaning about mathematical sophistication more complex. Similarly, the complexity of the problems posed in each of the studies is very different, and this has repercussions for the level of language used and thence the type of sociomathematical norms which will be established. It also makes the norms more difficult to identify. However, the norms in this study were consistently identifiable over three age groups at the secondary level.

It is interesting that, in the most established friendship groups (Year 10), negotiations of sociomathematical norms were found to be as equally identifiable as in the less established working groups (Year 7 and Year 9). Whilst Cobb *et al* assert that there is

mathematical specificity to any sociomathematical norms that are interactively established in groups in any setting, it may well be the case that these norms may also be context specific and therefore generate a need for groups to establish new taken-as-shared meanings in each of these contexts. Thus, established friendship groups are combining a mutually shared understanding of some established sociomathematical norms but, in a new mathematical context, are needing to generate and negotiate new norms. Since the study undertaken by Cobb *et al* was in a classroom where the teacher and class were undergoing a change in pedagogy and methodology towards social constructivism, it would be possible that the sociomathematical norms established in these conditions may not apply to a classroom where this mode of working is already an established norm. However, there is sufficient evidence in this small study to contradict this assumption. Indeed, a further sociomathematical norm was identified which I shall term *mathematical evidence*. This is demonstrated by the taken-as-shared meanings for the effective written communication of mathematical understanding.

Friendship groups appear to provide the necessary conditions for students to successfully challenge and justify ideas. The evidence to confirm Nelson and Aboud's (*ibid*) findings that friendships offer an environment in which learning leads to greater cognitive change for social situations may be transferable to mathematical learning. This is confirmed by Zarjac and Hartup (1997) who found that friends were better co-learners than non-friends. Whilst there is evidence in the Year 10 example in the study described here, the wider evidence from all three age groups confirms that friends are deferring to the more acceptable and efficient mathematical explanations.

REFERENCES

Adams, R and Allan, G (1998) *Placing Friendship in Context*, Cambridge, CUP

Allan, G (1989) *Friendship: Developing a sociological perspective*, Hemel Hempstead, Harvester Wheatsheaf

Allan, G (1996) *Kinship and Friendship in Modern Britain*, Oxford, OUP

Bell, S and Coleman, S (1999) The Anthropology of Friendship:enduring themes and future possibilities. In Bell, S and Coleman, S (Eds) *The Anthropology of Friendship*, London, Berg

Cobb, P (1995) Mathematical Learning and Small-Group Interaction: Four Case Studies. In Cobb, P and Bauersfeld, H (Eds) *The Emergence of Mathematical Meaning: Interaction in Classroom Cultures*, Hillsdale NJ, Lawrence Erlbaum Associates

Edwards, J (2003) *Mathematical Reasoning in Collaborative Small Groups: the role of peer talk in the secondary school classroom*, Unpublished PhD thesis, University of Southampton

Edwards, J (2004) Friendship Groups and Socially Constructed Mathematical Knowledge, *Proceedings of the British Society for Research into Learning Mathematics*, **24**(3), 7-14

Erwin, P (1993) *Friendship and Peer Relations in Children*, Chichester, John Wiley and Sons

Graded Assessment in Mathematics (1992), *Graded Assessment in Mathematics*, Walton-on-Thames, Nelson

Gottman, J and Parker, J (1986) *Conversations of Friends: speculations on affective development*, Cambridge, CUP

Nelson, J and Aboud, F (1985) The resolution of social conflict among friends, *Child Development*, **56**, 1009-1017

Noddings, N (1990) Constructivism in Mathematics Education. In Davis, R, Maher, C and Noddings, N (Eds) Constructivist Views on the Teaching and Learning of Mathematics, *Journal for Research in Mathematics Education*, Monograph 4, Reston, Virginia, NCTM

Pahl, R (2000) *On Friendship*, Cambridge, Polity

Rogers, P (1995) Putting Theory into Practice. In Rogers, P and Kaiser, G (Eds) *Equity in Mathematics Education: influences of feminism and culture*, London, Falmer Press

Schneider, B (2000) *Friends and Enemies: peer relations in childhood*, London, Arnold

Selman, R (1980) *The Growth of Interpersonal Understanding: developmental and clinical understandings*, New York, Academic Press

Yackel, E and Cobb, P (1996) Sociomathematical Norms, Argumentation, and Autonomy in Mathematics, *Journal for Research in Mathematics Education*, **27**, 458-477

Zarjac, R and Hartup, W (1997) Friends as Coworkers: research review and classroom implications, *The Elementary School Journal*, **98**, 3-13

THE USE OF A SEMIOTIC MODEL TO INTERPRET MEANINGS FOR *MULTIPLICTION* AND *DIVISION*

Marie Therese Farrugia

University of Malta

One important aspect of mathematics education is for teachers to share meanings for mathematical words with their pupils. I was interested in exploring how a meaning for a word may be rendered clear in primary mathematics classrooms, and in order to interpret 'clarity of meaning', I used a semiotic model which I developed by building on a model offered by Steinbring (1997, 2002). In this paper, I explain the development of the model. Using data I collected from a Grade 3 classroom (7 to 8-year-olds), I illustrate the possible application of the model by discussing sharing of meaning for multiplication and division in terms of semiotic chains.

INTRODUCTION

One important aspect of mathematics education is that pupils come to use and understand meanings for mathematical vocabulary. Mercer (2000) stated that teachers introduce their pupils to technical vocabulary by using the words in contexts that make their meanings clear. As part of a doctoral project I conducted in Malta, wherein I focussed on mathematical language, I wished to qualify what rendered meaning 'clear' as teachers attempted to 'share' meanings for words with their pupils.

My assumption regarding the teaching/learning process was that pupils appropriate a meaning for words as they participate (overtly or silently) in the discourse that is particular to the mathematics classroom. Hence, I considered 'meaning' in the sense of how a word was used in relation to other words and pictures and/or notation. Hence, I interpreted 'shared meaning' in terms of a similarity between statements offered in the classroom by the teacher, and explanations offered by the pupils afterwards. I did not expect pupils' expression of meaning to be an exact replication of what had been said in the classroom, but allowed for some variation in the way meanings were expressed. As stated by Chapman (2003), similar ideas can be expressed through different semantic terms.

I felt the need of an analytic tool that would allow me to discuss meaning and in particular wished to consider more theoretically the notion of 'clarity' of meaning. I turned to a consideration of sign systems or semiotics, since this was in line with the social perspective that I adopted in my study. In this paper, I explain my development of a semiotic model and offer illustrations of its use. I reflect on 'clarity' by using the model to interpret instances of successful and unsuccessful sharing of meaning.

THE DEVELOPMENT OF A SEMIOTIC MODEL AS A TOOL

Generally speaking, a 'sign' is something that stands for something else in the sense of *X represents Y* (Tobin, 1990). Various things can be considered signs, including art, writing, diagrams, pictures, counting systems, algebraic symbols and even language itself (Vygotsky, 1981). Steinbring (1997) considered numbers as mathematical signs and stated that meanings for mathematical concepts emerge through an interplay between signs, or symbols, and objects, or reference contexts. He suggested that this relationship could be represented by the following diagram:
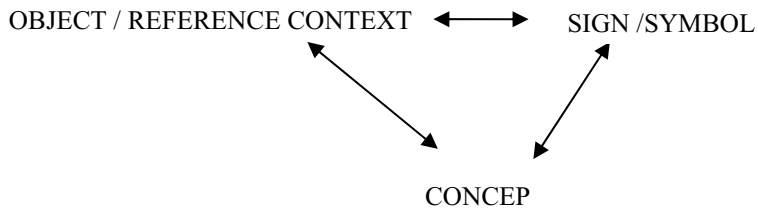
OBJECT / REFERENCE CONTEXT ⟷ SIGN /SYMBOL

CONCEP

Figure 1 Steinbring's (1997, 2002) epistemological triangle

As an example of a triad, Steinbring (2002) offered '3' as a sign/symbol, diagrams of three apples / balls as a reference context and 'elementary number concept' as the third component. In another example, Steinbring gave the respective elements as: $\sqrt{2}$, a unit square with a diagonal marked in, and 'aspect of the concept of real numbers' (Steinbring, 1997). Steinbring (*ibid*) considered that the notation functions as a sign because it represents the object in some respect. For example, the symbol 3 refers to the numerosity of the set of balls and not to say, their colour or shape. For the benefit of young children, the reference context is often a real life context or a picture, but Steinbring (1997, 2005) stated that the empirical character of knowledge can be increasingly replaced by diagrams or other sign systems in order that relational connections are set up. Furthermore, Steinbring (2002) suggested that a sequence of 'triangles' can be drawn up to illustrate the development of a child's interpretations. Figure 1 above served as a starting point for a model I devised, shown below in Figure 2.
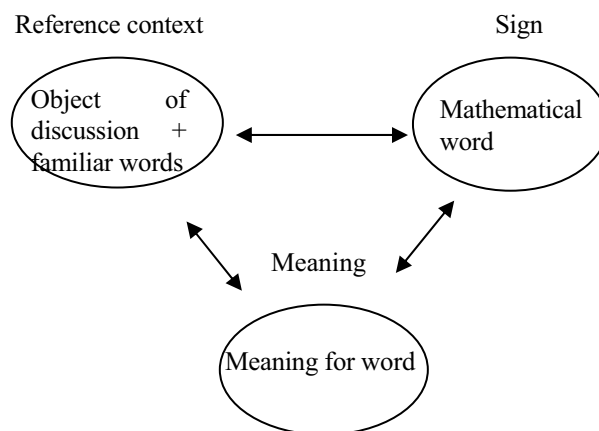
Reference context   Sign

Object of discussion + familiar words ⟷ Mathematical word

Meaning

Meaning for word

Figure 2 My own semiotic model

I retained the label 'sign' but considered that it might also be a mathematical word, since in my study I had a particular interest in mathematical vocabulary. I replaced Steinbring's label 'concept' with the word 'meaning'. This was because while Steinbring had considered number relationships, I wished to consider words that denoted a variety of notions: properties (e.g. *irregular*), actions (e.g. *measure*) and even words that served a referential role (e.g. *x-axis)*. I do not normally refer to these as 'concepts', a term I reserve for relationships such as multiplication (which in fact I discuss in this paper). I considered a reference context to incorporate both an object and 'familiar words'. Wertsch (1985) explained that any situation, event or object has many possible interpretations and speech serves to impose a particular interpretation. Hence, I suggest that an object serves its purpose in the development of mathematical ideas thanks to what is rendered salient through language. So for example, when handling a 10 cents coin, a teacher might use language to direct attention to the number on a coin in order to lend meaning to the word *value*. The choice of this language can be contrasted to other alternatives that would draw attention to the images on the coin, its thickness, the material it is made of and so on. Hence, I considered the reference context to be an object *together with* accompanying language.

RESEARCH METHOD

The general design of my data collection was to observe and video-record a number of lessons (34 hours in all) in two primary school classrooms (Grades 3 and 6, ages 7-8 and 9-10 respectively, in a girls' school). I focused on parts of the lessons where topic-related vocabulary was used, transcribing these parts and noting how the words were introduced and used. After the lessons, I interviewed six pupils per class, per topic, regarding their understanding of the selected mathematical words. In this paper, I discuss an aspect of the Grade 3 topic 'Multiplication and Division'. (At this point I must mention that the lessons were carried out in the participants' second language, English. Although this situation constituted an important part of my main project, it is beyond the scope of this paper to reflect on this aspect. In this paper I focus on the general approach taken by the teacher. Furthermore, during the interviews, the pupils tended to code-switch between Maltese and English since, in Malta, mathematical vocabulary is retained in English. Again, I will not discuss this aspect here, but consider meaning as expressed through the two languages. For the benefit of a non-Maltese reader, I have translated Maltese speech and printed it in a **bold** font when presenting transcriptions).

APPLICATION OF THE MODEL TO INTEPRET SHARING OF MEANING

Multiplication and division as procedures

The Grade 3 teacher reported that the pupils had learned the multiplication tables in the previous Grade. Indeed, the girls could already recite these in the form of, say, "one three is three, two threes are six, three threes are nine …" or "three, six, nine …" while opening up fingers, one at a time. The teacher's aim was for the pupils to now apply the tables to 'situations'. My observations and interviews indicated that over the week, the pupils came to consider the words *multiply*, *multiplication* and *times* to be closely associated, and similarly the words *divide* and *division* (all words were new to the pupils except *times)*. These groups of words were respectively used in relation to the notations $m \times n$ and $x \div y$, the solutions of which were found by reciting the tables. For example, the answer to 4×3 (or 3×4, the teacher and pupils used these interchangeably) was recognised as "four threes are TWELVE"; for $12 \div 3$, the answer was found as "FOUR threes are twelve". Each time, four fingers were opened up. I concluded that the girls had learnt a meaning for multiplying and dividing as procedures. However, I was also interested in examining whether *concepts* for multiplication and division had been successfully shared with the pupils, in the sense of multiplication as repeated addition of similar sets of items, and division as repeated subtraction or formation of equal groupings. I found that while the pupils expressed appropriate meanings for multiplication, this did not appear to be the case for division. I consider each in turn.

Successful semiotic chaining for *multiplication*

The following excerpts illustrate that the pupils I interviewed recognised multiplication as a relationship between a number of similar sets and the 'size' of each set. For example:

> Sandra: **In** multiplication, **you don't keep doing** six plus six, plus six (*opens three fingers, one at a time*). You just multi-, you just multi- … three tim- … multiply by six and you get the answer.

> Kelly: (*Points to a textbook picture of two three-legged monsters*). Now here is two monsters. And they have three legs [each]. Now you to find, to times … because there are two monsters, and then you count the legs (…) and you write them here (*touches the notation she herself had written in pencil $2 \times 3 = 6$*) and … then you write the answer.

I suggest that the pupils' success in offering appropriate explanations was due to the fact that this particular meaning for multiplication as repeated addition had been 'clearly' expressed through the classroom interaction. For example:

> (*The class is looking at a textbook page showing monsters with three legs each*).
> Teacher: Every monster has three legs each. Now in the first set of monsters there are four of them. Now each one has three legs. So I have 3 legs, 3 legs, 3 legs, 3 legs. (*Opens 4 fingers in turn*).I can count: 1, 2, 3, 4, 5, 6, 7 … Or else we said in our first lesson we can…? What can we do?

	Petra?
Petra:	Three times four.
Teacher:	Very good. We can MULTIPLY instead of having a repeated addition. Instead of adding three plus three, plus three, plus three (*writes notation 3 + 3 + 3 + 3 on the board*), what can I do? I can group all this (*draws a large oval around the notation*)… Each monster has three legs. So instead of adding three for four times, what can I do?
Jane:	Three times four.
Teacher:	I multiply three by four. (*Writes 3×4 on the board*). And how many legs is that?
Pupils:	Twelve!
Teacher:	(*Completes equation on board: 3×4 = 12*). We could count the legs, but I would like to see this multiplication (*touches notation on board*).

The initial reference context consisted of monster pictures and the language 'Each monster has three legs', 'three legs, three legs …'. The language used led the pupils to give a quantitative interpretation to the pictures, unlike other possible statements such as 'The monsters are happy' or 'Are they wearing shoes?' The language drew attention to similar groupings, hence establishing a meaning for repeated addition as similar groupings. This was represented by the sign $3 + 3 + 3 + 3$. This same notation was then used as part of a new reference context. The language used now drew attention to the four-fold presence of the number 3 and suggested an interpretation for multiplication notation as an alternative to addition notation. This was offered by way of the words '*instead of*', '*three for four times*' and '*repeated addition*'. A chain of meaning can be illustrated as in Figure 3 overleaf. Although the recitation of the 'table' is not evident in the above transcript, I also include the procedure of multiplying in the diagram in order to show how the word *multiply* (which was closely associated with the word *multiplication* by both teacher and pupils on several occasions) came to express more than the simple recall of the Tables. Throughout the week, I noted several instances for which part, or all, of a similar diagram could have been drawn, albeit using different items, numbers and variations of language. For this and other examples, I considered that meaning was rendered clear thanks to the 'proximity' of the language and the pictures/notation, in the sense that to what the language was referring was evident. Hence, taken together, the language and objects offered a supportive reference context thus 'gluing' (Hewitt, 2001) the words *multiply* and *multiplication* with the mathematical idea of repeated addition.

A concept for division as the formation of equal groups

One interpretation of division is the formation of sets of a given quantity. A diagram parallel to Figure 3 is possible for x items being grouped into sets of y. The diagram would include the notation $x − y − y − y$ …. and $x ÷ y$, supporting language ('y buns in each bag', 'groups', 'How many in each set?', 'repeated subtraction' etc.) and the words *divide / division*. The teacher did, in fact, use the words *groups/grouping*
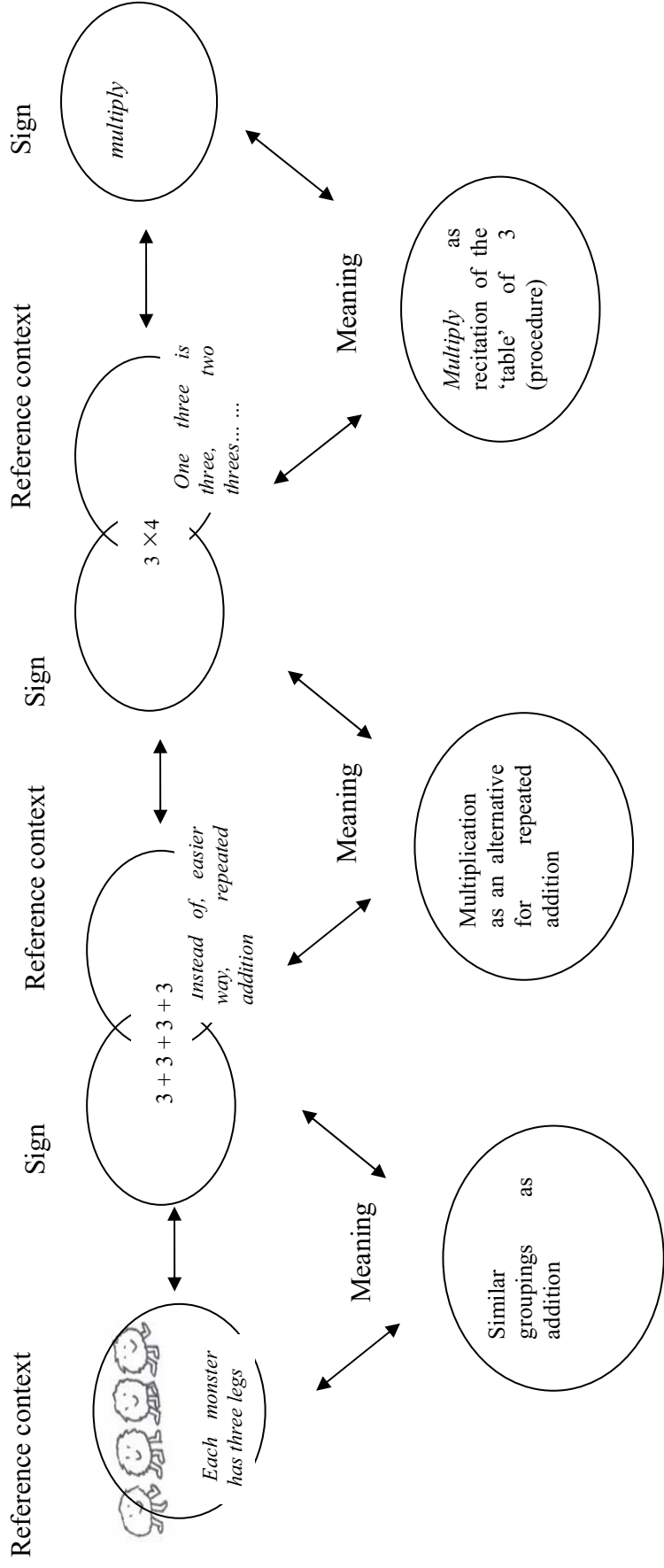
Figure 3 Semiotic chaining for multiplication (diagram adapted from Merttens and Kirkby, 1999)

several times, and also used the expression *repeated subtraction*. However, the pupils I interviewed at no time gave an explanation for division that considered the formation of equal sets of items. Rather, I noted three types of explanations: *division* as a procedure, *grouping* as a sharing activity, and 'repeated subtraction' as $x - x - x - x \ldots$ The following excerpts illustrate these types of explanations.

> Ramona: **You'll have** eleven, **and you do** divide b- b- divide by three equals … **and the** answer **comes** smaller.

> Maria: [Grouping is] **when you've got eight dolls and you share them. And so I'll have to share them between two [girls] and they have to be enough [to go round] for everyone.**

> Sandra: [Repeated subtraction] **means** if you have three minus three, minus three, minus three … (*trails off*).

By examining the classroom interaction, I noted that unlike multiplication situations, for which a common element across all the situations presented had been a repeated quantity, division situations presented were very diverse. Furthermore, the reference contexts offered by the teacher appeared not to be supportive enough due to the choice of language used in conjunction with pictures/notations. I give three examples as illustrations.

One activity carried out was related to a textbook diagram showing a kangaroo jumping in threes on a horizontal number line, with each hop marked with an arc. The first exercise had focused on multiplication ("find what number the kangaroo lands on if he jumps 4 hops"), while the second exercise was intended as division, where pupils had to find how many hops were needed to land on 9, 15 etc. As this latter exercise progressed, the teacher wrote the following notation on the board:

11. $9 \div 3 = 3$ hops

12. $30 \div 3 = 10$

13. $21 \div 3 = 7$

She then explained to the class:

> In the division, the number look, becomes SMALLER. See? (*She runs her finger down the quotients column - 3, 10, 7 - then up the dividends column 21, 30, 9*). It always becomes smaller because we are dividing, grouping … It is a repeated subtraction.

I myself was able to interpret the gesture to mean that say, 3 was less than 9, 10 less than 30 etc. The teacher later told me that she had mentioned subtraction in the hope that the pupils would associate subtraction and division in that, as she stated, they both 'made smaller'. However, I suggest that the link between subtraction and division was not expressed clearly since it was not perceptually evident to what the teacher's language referred. The repeated subtraction notation $9 - 3 - 3 - 3 = 0$ was not used nor did the situation imply anything 'taken away'. The only 'repeated' things were the divisor 3 and the symbols $\div$ and $=$ present in each example. As the teacher

moved her finger down and up the numbers, no obvious pattern of something getting smaller could be perceived. Although 'groups' of 3 were implied in the diagram, the kangaroo was shown jumping 'up' the number line; a more helpful representation for division as repeated subtraction would have been the kangaroo jumping 'down' i.e. right to left.

In another exercise, the pupils worked out how many 5p coins were needed to buy stamps of 30p, 45p etc. The first example dealing with a 30p stamp was shown on the book as '30p ÷ 5p = 6'. The teacher encouraged the girls to use the same format for the other examples, and to use the 'Tables' to find the answers. The teacher talk included "we're going to give it to the post office in 5 pence coins", "we are grouping the 5 pence coins". Although the teacher may have wished to infer that the five pence coins embodied a 'group' of five 1p values, no items had actually been grouped.

One situation that offered potential to interpret division as formation of groups was a word problem as follows: "*Mick has 20 cans to pack. 5 cans go in each box. How many boxes?*" The language of the story sum and an adjacent picture showing the situation gave a sense of the action taking place. The classroom interaction proceeded as indicated below.

Teacher:	I'm going to act it out (...) How many cans has he got? (*makes a gesture indicating putting things together and putting them aside*).
Pupil:	Twenty.
Teacher:	Twenty. Now they do not fit all in one box. He takes five of them and puts them in a box (*Place hands close together and mimics putting something aside. This gesture is done four times as the teacher talks*). [So] Five in one box, another five in a box, another five in a box, another five in a box. We want to know how many boxes we need. Annemarie?
Annemarie:	(*Silent*).
Teacher:	(*Repeats above explanations and gestures*). What is happening here?
Annemarie:	Grouping. [NOTE: here the pupil is prompted the teacher's gesture, which she always used when uttering the word "grouping"].
Teacher:	How many boxes do I need?
Petra:	Four.
Teacher:	Four. How did you work that out?
Petra:	Five, ten, fifteen, twenty (*opens four fingers out in turn*).
Teacher:	[So] we already know the answer. Now I would like to work it out with a 'statement' and a division (*touches the 'statement' and notation '45 ÷ 5' had been written on the board for a previous, but very different, story sum*).
Nadia:	"Five cans equals one box; twenty cans equals how many?"
Teacher:	(*Writes Nadia's suggested 'statement' on the whiteboard as shown*:

    5 cans = 1 box
    20 cans =   ?

What do I write here? (*Touches the board underneath the 'statement'*)

Rita:	Five division by twenty. [Note: it was common practice in the classroom to use the expression *division by* instead of *divided by*].
Teacher:	You can't divide five by twenty! Can you turn it the other way round?
Rita:	Twenty division by five.
Teacher:	(*Writes 20 ÷ 5 on the board*). And how many boxes is that?
Melissa:	Five, ten, fifteen, twenty (*opens up fingers on one hand as she counts*). Four.
Teacher:	Four. Then I write my answer. (*Completes equation on whiteboard as shown*):

$$20 \div 5 = \quad 4 \text{ boxes}$$

Annemarie and Melissa found the solution to the problem by counting up in fives; Annemarie appeared to be prompted by the teacher's gesture, and Melissa by the notation. However, I felt that the various aspects of the packing situation were not focused on explicitly by the teacher in their relation to the division notation, in the sense of the original set to be acted on being represented by 20, the size of the group represented by the 5, the relation between the action of packing and the division sign, and the quotient being the already-known solution, four. Perhaps the teacher could have linked more specifically the idea of the repeated formation of groups with 'dividing' by writing out the notation as she referred to the various aspects of the situation, as she had done for multiplication. I felt that the writing of the 'statement' and subsequent manipulation of the notation *per se* hindered the setting up a link between the action/picture of grouping and the division notation.

Through these and other examples, I concluded that the various reference contexts utilised for division were not supportive enough to enable the pupils to link the division notation with the formation of equal groupings. Rather, when asked for an explanation, they 'fell back' on possibly more familiar ideas such as the procedure and the action of sharing, or drew on their knowledge of repeated addition as $n + n + n \ldots$ to suggest that repeated subtraction meant $x - x - x - x \ldots$

CONCLUSION

The Grade 3 pupils appeared to appreciate multiplication as repeated addition, but not division as repeated subtraction. Assuming that clarity of meaning as expressed in the classroom had some bearing on the pupils' ability to offer appropriate explanations, I attempted to qualify what had rendered meaning 'clear'. I considered that the objects utilised together with any accompanying language constituted a reference context, and concluded that a supportive reference context was one wherein there appeared to be a 'proximity' between the two. In such instances, it was evident to which aspect of the object the language referred. I considered that it was this proximity that rendered meaning clear. On closer examination of the classroom data, I found that while proximity was evident in the multiplication situations, this was not the case for division. I was able to apply this view of clarity to other data I collected, a view that

may be useful to explore further as part of reflections on the teaching of new mathematical vocabulary.

REFERENCES

Chapman, A. P. (2003). *Language Practices in School Mathematics: A Social Semiotic Approach*. Lewiston: Mellen Studies in Education.

Hewitt, D. (2001). 'Arbitrary and necessary: Part 2 Assisting memory'. *For the Learning of Mathematics 21*(1), 44-51.

Mercer, N. (2000, first published 1995). *The Guided Construction of Knowledge*. Clevedon: Multilingual Matters.

Merttens, R., & Kirkby, D. (1999). *Abacus (Mathematics Scheme)*. Oxford: Ginn.

Steinbring, H. (1997). 'Epistemological investigation of classroom interaction in elementary mathematics teaching'. *Educational Studies in Mathematics 32,* 49-92.

Steinbring, H. (2002). *What makes a sign a mathematical sign? - An epistemological perspective on mathematical interaction.* Unpublished paper presented to the Discussion Group on Semiotics in Mathematics Education at the 26th PME International Conference, England, University of East Anglia, July 21-26, 2002.

Steinbring, H. (2005). *The Construction of New Mathematical Knowledge in Classroom Interaction.* New York: Springer.

Tobin, Y. (1990). *Semiotics and Linguistics*. Essex: Longman.

Vygotsky, L. S. (1981). 'The instrumental method in psychology'. In J. Wertsch, *The Concept of Activity in Soviet Psychology.* New York.: M.E. Sharpe.

Wertsch, J. (1985). *Vygotsky and the Social Formation of Mind*. Massachusetts: Harvard University Press.

"WHY SHOULD I IMPLEMENT WRITING IN MY CLASSES?" AN EMPIRICAL STUDY ON MATHEMATICAL WRITING

Marei Fetzer

J. W. Goethe-University Frankfurt/M., Germany

An empirical study on mathematical writing in primary education serves as the basis for this article. Students wrote about their problem-solving process, they presented their approaches and negotiated alternative ways of proceeding. Here the focus is put on the emergence of interactional conditions that enable mathematical learning if students discuss alternative approaches based on their written works. Three dimensions are called in to describe how optimized learning conditions emerge within the interaction of the mathematics classroom: the aspect of participation, thematic development and the argumentative aspect. Their interplay provides an enhanced set of terms to approach the aspect of learning in the context of mathematical writing.

INTRODUCTION

"Why should I implement writing in my classes?" (D. Miller 1991, p.518). This question was asked 15 years ago by Diane Miller. In her article she gives an answer similar to those Morgan (1998), Pugalee (2005), Lesser (2000), Gallin and Ruf (Ruf/Gallin 2003) or the NCTM standards (2000) offer today: Writing should be an integral part of the mathematics classroom, because students as well as teachers benefit from this way of working. From a scientific point of view this empirical result is to be appreciated, but at the same time it is unsatisfying (see e. g. Borasi/Rose 1989, p.349). There remains the desire to understand and explain what happens if students write about mathematical concepts and individual problem-solving processes and if they read their work and discuss alternative approaches. How and why does writing contribute to mathematical learning? Where are the positive influences of writing to be situated within the process of learning?

"How does writing improve learning?" (D. Miller 1991, p.516). To me, this is the essential topic to be dealt with. How do situations emerge that make learning possible? What interactional conditions enable mathematical learning? In this article I try to approach these questions. First I am going to give some information concerning the empirical study my work is based on. Afterwards I am going to introduce a set of terms. Finally I am going to present selected empirical results.

THE EMPIRICAL STUDY

In the empirical study, mathematic classes were observed within one course from the first to the third grade. Special emphasis was laid on writing lessons. In order to get hold of the process of writing on the one hand, and to approach aspects of reading,

presenting and discussing on the basis of the written works on the other hand, my empirical study was designed in two phases: the writing phase and the publishing phase. Within the writing process students "externalize" (Bruner 1996) their problem-solving process in a written form. These written works serve as a basis for the subsequent whole-class publishing situation. In this phase several children present their way of proceeding when working on the given task on the board. Alternative approaches are discussed. During the publishing phase all students have their individual works at hand all the time. They might have a quick glance at it at any time.

Most approaches to writing in mathematic classes focus on the products of students' writing (see e. g. Selter 1994; Ruf/Gallin 2003; Pugalee 2005; Morgan 1998; Borasi/Rose 1989; Fetzer 2003b; Krummheuer/Fetzer 2005). It is assumed that mathematical learning takes place within the writing process. However, doing research on the publishing of the works and the discussion based on the written products is widely neglected. As a consequence my research activities concentrate on these latter aspects (amongst others). In this article whole-class publishing situations are the focus of interest.

If the emphasis is put on interactional situations in the publishing phase it becomes evident that research cannot be restricted to the analysis of the students' written products. Other aspects gain weight: How do students explain their proceeding? How do they put forward arguments? How do they refer to their own written work and the board? In order to reconstruct how processes of interaction emerge within the publishing phase, 32 mathematics writing classes were videotaped during a three year period. Afterwards transcripts showing verbal and nonverbal aspects as well as the current writings on the board were produced.

Methodologically, processes of interaction are approached by an analysis of interaction. Thus the emerging interactional process can be reconstructed step by step. The interactional analysis is a method derived from conversational analysis (see Eberle 1997; ten Have 1999). In the context of my study I apply the method in the same manner as introduced by Krummheuer and Naujok (1999).

TERMINOLOGICAL BASIS

"How does writing improve learning?" (D. Miller 1991, p.516). How do situations emerge that enable mathematical learning? In order to approach these questions I now introduce a set of terms. In so doing I outline my theoretical framework. In addition I present terms I developed within my empirical study.

To me referring to M. Miller (1986), learning is a matter of participating in interactional processes. Students learn mathematics by being part of and taking part in the ongoing argumentative processes of mathematics classes.

In order to understand interactional processes in class, I refer to three aspects: The participative aspect, thematic development and argumentation. These aspects have been developed empirically (see Fetzer 2006a; Krummheuer/Fetzer 2005). They help to capture interactional processes in the mathematics classroom. Each of these dimensions is explained in the following.

Participation

Participation is understood as the students' or teacher's participation in classroom interaction. Participating in this educational context can be distinguished as 'taking part' on the one hand and 'being part of' on the other hand. Taking part is an active form of participating, whereas being part of is a rather receptive one. However, a receptive participant of the classroom interaction may change her/his status of participation and take action (see also Fetzer 2006a).

The following two examples, both taken from a whole-class publishing phase, are meant to explain some terms developed within the empirical study. Just before the first episode begins (transcript 1), Benno has explained on the board how he proceeded in working on the given task. When Benno calls Sonja's name she asks: "How'd you get the twelve if you (incomprehensible) the two?" Thereupon Benno begins to explain his proceeding again.

Person	Aktivität	Activity
Benno	Sonja-	Sonja-
Sonja	Wie kummsch n da(nn) auf die **zwölf**	How'd you get the **twelve**
<Sonja	wenn du die zwei (unverständlich)-	if you (incomprehensible) the two-
<Benno	Von **elf** Zentimeter	Starting with **eleven** centimeters
	zeigt erst auf die 11, dann auf die cm	*points at 11 first, then at cm*
	5cm+6cm=11cm	***5cm+6cm=11cm***
	8mm+4mm=12mm	***8mm+4mm=12mm***
	11cm+12mm=12cm2mm	***11cm+12mm=12cm2mm***
	habisch-	I have-
	zeigt auf die 12 (mm)	*points at the 12(cm)*
	5cm+6cm=11cm	***5cm+6cm=11cm***
	8mm+4mm=12mm	***8mm+4mm=12mm***
	11cm+12mm=12cm2mm	***11cm+12mm=12cm2mm***
	also **elf** Zentimeter hab isch schon-	well, I've already got **eleven**-

Transcript 1: How'd you get the twelve (See transcription rules, below)

Regarding the organization of "turn-taking" (Sacks 1996), how does it happen that Sonja takes the role of the current speaker? Benno calls Sonja by name; he selects her as next current speaker. Sonja accepts the turn and starts speaking. This way of taking a turn after being addressed personally I call *"accepting a turn"*.

In the following example (transcript 2) the second way of turn-taking I could identify within my studies is introduced. The episode occured a couple of minutes earlier. Sonja is the one who presented her approach to the task on the board. Sabina expresses her embarrassment: "Somehow I don't get it." Sonja starts explaining

again: "I have five, that made five up there." Then Benno says: "That was eight cen millimetres."

Person	Aktivität	Activity
Sabina	Ich kapier des irgendwie net-	Somehow I don't get it-
Sonja	*Wendet sich Richtung Tafel, nachdrücklich* Isch hab- **fünf**/+ *deutet dabei mit re gestreckten Arm auf Tafel, nimmt Arm runter*	*Turns in the direction of the board, with emphasis* I have- **five**/+ *points to the board with stretched arm, lowers the arm*
<Martina	*Meldet sich*	*Puts up her hand*
<Sonja	# des ergabs ja oben fünf\ *geht beim Sprechen zum Tisch, greift ihr Blatt mit beiden Händen ohne es vom Tisch zu nehmen und schaut darauf*	# that made five up there\ *while talking she walks to her desk, grabs her sheet with both hands without taking it from the desk and looks at it*
Benno	Des war'n **acht** Zen *leiser* Millimeter-+	That was **eight** cen *in a lower voice* millimeters-+

Transcript 2: That was eight millimeters

Benno did not accept a turn as Sonja did in the first example. He was neither addressed nor was his name called. However, he seems to grab the next speaker's role. He simply takes hold of it without being called on. If the current speaker does not select the next speaker, if there is no change in turn-taking intended by him or her, but someone gets hold of the turn nevertheless, I call this way of turn-taking "*seizing a turn*".

The third way of taking a turn is to "*pick it up*". Time and again the current speaker leaves an opening and thus offers the next speaker's role to somebody else, but does not address anybody specifically. He or she 'invites' other members of the interaction to take over. There is the possibility to "*refuse*" an offered turn. (See also Fetzer 2006b).

Thematic development

In introducing terms concerning thematic development I want to describe the specific interactional situation of the publishing phase. During the writing process the students produced written works. While the publishing phase proceeds, each student has his or her own work at hand and may glance at it at any time. At the same time he/she can see the board. Accordingly every student has access to two medially graphic elements (see Oesterreicher 1997; Oesterreicher/Koch 1985; Fetzer 2003a) at a time: his/her own written work and the board. Both graphic elements may be either "*consistent*" or show thematic breaks and be "*inconsistent*" (Fetzer 2006a).

Argumentation

In order to understand the nature of logical processes in publishing situations, I use Toulmin's approach to argumentation as a tool of analysis (Toulmin 1969; see also Fetzer/Schreiber/Krummheuer 2004). According to this approach, utterances are not

seen from the perspective of what the speaker might have intended to express or what he might have meant. Instead, the aim of the analysis is to reconstruct the function which an individual's utterance fulfils within the argument. Toulmin's analysis of argumentation helps to determine those functions.

Based on Toulmin's work, the shortest possible argument consists of the two elements data and conclusion. The conclusion is the claim that needs to be established when it is challenged; it has to be shown that it is justifiable. The data is our personal knowledge, the facts we appeal to as a foundation for the claim, the ground we produce as su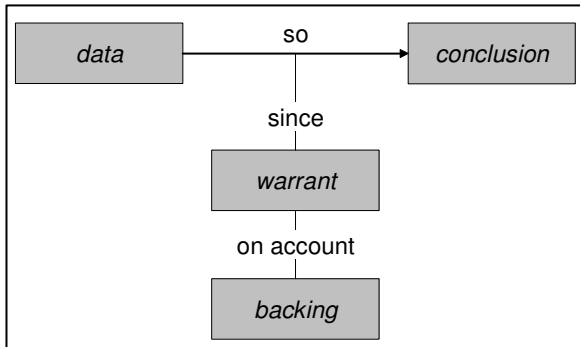pport for the original assertion. Such a 'simple' argument can be summed up as follows: Data *D* is the basis *so* the conclusion *C* can be established. An argumentation gets more complex if warrants are included. Warrants are general, hypothetical statements which can act as bridges and indicate the bearing on the conclusion on the data already produced. You can get from *D* to *C since* the warrant *W*. Arguments get even more complex, if backings of warrants are offered. Warrant *W* is acceptable in general *on account* of backing *B* (Toulmin 1969, p.94ff.). Toulmin put the structure of arguments into a graphic layout shown in the figure (1).

Figure 1: Toulmin's layout

Most of the time in the mathematics class primary students may get along producing *simple arguments* consisting only of data and conclusion. But sometimes simple arguments do not meet the requirements of the situation anymore. For example the teacher demands further explanation to the statement 3+7=10. Warrants might be put forward: I did it by the use of wooden pearls; I counted; Joe figured out the same result. If more *complex arguments* are expected to fulfil the interactional demands, I describe the interactional situation as "*argumentatively condensed*" (Fetzer 2006a/b).

EMPIRICAL RESULTS

"How does writing improve learning?" (D. Miller 1991, p.516). Now that the terms needed to approach this question have been introduced I am going to describe how interactional situations in publishing phases emerge that enable mathematical learning. To illustrate I start by considering the example introduced above (transcript 2).

Example

The task was to lengthen a given line by 6cm 4mm (fig. 2, left). Sonja is the first child to present her approach on the blackboard (fig. 2, right). After Sabina's question Sonja starts explaining again. Then Benno interferes. Without being asked
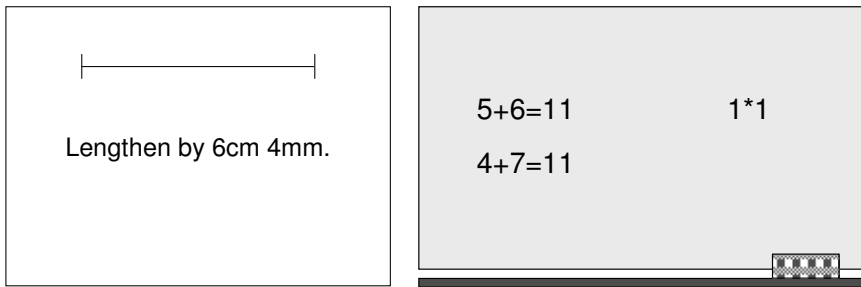
Lengthen by 6cm 4mm.

$5+6=11$ $1*1$

$4+7=11$

Figure 2: Task (left), current writing on the board (right)

he gets hold of the current speaker's role: "That was eight millimetres." Regarding the dimension of participation he seizes the turn. How does it happen? What conditions make Benno take over an active role in the interaction and seize the turn? Remember the setting of the publishing phase: Benno has access to the blackboard and his own written work at the same time. Obviously there are inconsistencies between both graphic elements: On the board it says "7", whereas Benno's work does not contain the number 7 at all. Instead, his written product shows an eight, but not in the sense of a number but as a measured value: 8mm. Benno recognizes these thematic breaks. He identifies the two graphic elements to be inconsistent and as a consequence he seizes the turn and contradicts: "That was eight millimeters." At this point the third dimension, the aspe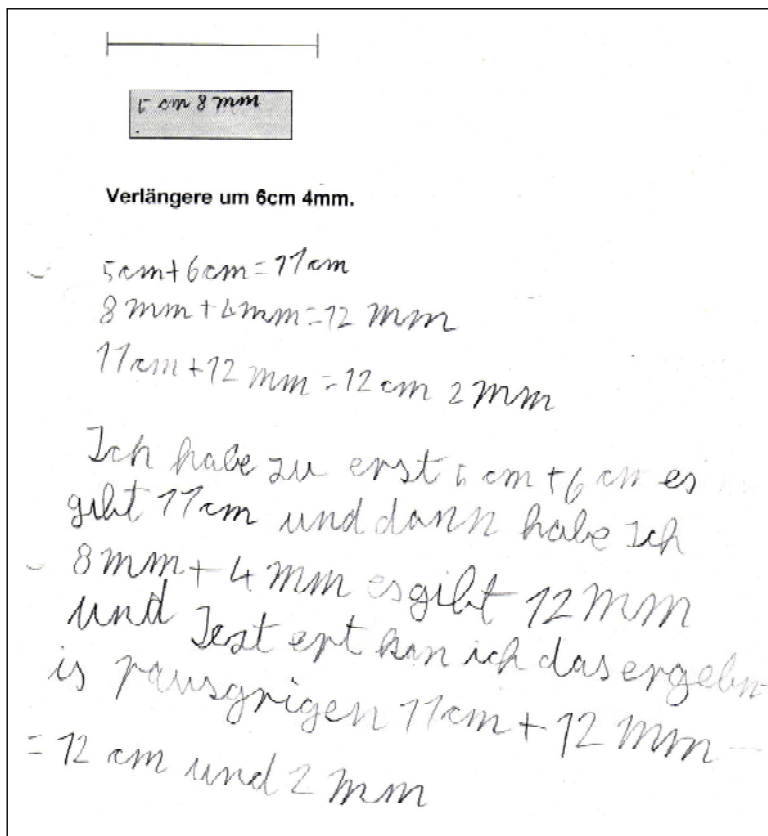ct of argumentation, gains relevance. With Benno contradicting he makes the inconsistencies between the graphic elements 'work' and 'board' explicit to all members of the interaction. He points out the two senses of interpreting the digits, either as a number or as a measured value. Besides he alludes to the different results of measurement (7(mm) and 8mm). With the inconsistencies being explicit and accessible to all members of the interaction, Sonja can't continue her explanation. There arises the need to negotiate those aspects Benno referred to. Attempting to explain what she did, Sonja started off with a simple

Figure 3: Benno's work

argument consisting of data and conclusion: "I have five, that made five up there…" This argumentative level does not meet the requirements of the interaction anymore. The interactional situation condenses argumentatively. Concerning the aspect of learning this is the crucial impetus. In order to take part in the process of interaction

current speakers need to put forward more complex arguments. Warrants are required to bridge data and conclusion. Eventually even backings might be demanded. Such argumentatively condensed situations urge the presenting child to re-think her approach (Sonja). Besides, for the student who triggers the argumentatively condensed situation the conditions for learning are beneficial (Benno). Those who are taking a receptive role at the moment are required to contribute to the process of argumentation. Finally, there are those members of the interaction that remain 'quiet'. They might profit by hearing the complex arguments put forward.

This example shows how the emergence of interactional conditions that enable mathematical learning can be described by the synopsis of the three dimensions participation, thematic development and argumentation. In the given example one student seizes the turn when identifying inconsistencies between both graphic elements. The interactional situation becomes argumentatively condensed.

Summary

What interactional conditions enable mathematical learning? How do argumentatively condensed situations emerge? Analyses of numerous episodes of interaction in publishing phases reveal the following empirical results.

Regarding publishing phases there are two conditional settings that contribute to the emergence of argumentatively condensed situations in the mathematics classroom:

• If students identify *inconsistencies* between the own written work and the blackboard and thereupon contribute to the process of interaction actively, argumentatively condensed situations emerge. In this context it does not matter if the students accept, pick up or seize the turn.

• If students contribute to the process of interaction by *seizing the turn*, they evoke the emergence of argumentatively condensed situations.

In both cases students need to create rather complex arguments in order to participate actively in the process of interaction. To fulfil the interactional requirements arguments put forward need to exceed the basic structure of data and conclusion. The interactional conditions of learning optimize.

"How does writing improve learning?" (Miller 1991, p.516). Especially within the *publishing* phase argumentatively condensed situations emerge. Brief designations of the result of a mathematical task as well as the answer "I've got the same" are insufficient and do not meet the interactional demands. Instead, explaining and negotiating are appropriate activities to contribute to the process of interaction. Looking at the board one second and glancing at ones own written work the next second can serve as a basis for an attitude of "I did it differently". The possibility of relating both graphic elements enhances the chance to *identify inconsistencies* between work and board. At the same time the coinstantaneous access to both graphic elements provides self-confidence and thus supports *seizing a turn*. Working

with writing in the mathematics classroom turns the well-known "I've got the same" into "I did it differently", "I did not get it, please explain again", "How did you proceed?" Argumentatively condensed situations can be regarded as optimised learning conditions for all members of the interaction. Those who think about a coherent and convincing explanation will benefit as well as those who listen to the arguments presented. Students who contribute actively to the emergence of complex arguments will profit as well as children who take a receptive role.

Finally, I focus the title of this article and return to the opening question: "Why should I implement writing in my classes?" (Miller 1991, p. 518).

Based on the results of my empirical study I specify the answer offered in the beginning. Mathematical writing should be implemented in classes because, especially during the publishing phase, chances are good for the emergence of argumentatively condensed situations. If students present their written works, if they explain their proceedings, if they discuss different approaches and have access to their work and the board, optimised learning conditions are likely to emerge.

TRANSCRIPTION RULES

The first column indicates the names of the interacting persons. The second and third column give the verbal (regular font) and non-verbal (*italic font*) actions in English and in German.

/ - \	Rising, even, falling pitch.
bold	Accentuated word.
s p a c e d	Spoken slowly.
(incomprehensible)	Incomprehensible utterance.
+	The indicated way of speaking ends at this symbol.
#	There is no break, the second speaker follows immediately from the first.
<M four five six\	Indicates where people are talking
<S five	at the same time.

REFERENCES

Borasi, R. & Rose, B. J. (1989). Journal Writing and Mathematics Instruction, *Educational Studies in Mathematics, 20 (4),* pp. 347-365.

Bruner, J. (1996). *The culture of education* (Cambridge, MA, Harvard University Press).

Eberle, T. S. (1997). Ethnomethodologische Konversationsanalyse, in: Hitzler, R. & Honer, A. (Eds.) *Sozialwissenschaftliche Hermeneutik* (Opladen, Leske + Budrich).

Fetzer, M. (2006a). *Empirische Studie zur Entwicklung von Elementen einer Interaktionstheorie grafisch basierten Lernens – Schreibanlässe im Mathematikunterricht der Grundschule*, Unpublished Ph.D. dissertation, J.W. Goethe-Universität F. a. M..

Fetzer, M. (2006b). Veröffentlichen im Mathematikunterricht – ein Beitrag zu einer Interaktionstheorie grafisch basierten Lernens, in: Jungwirth, H./Krummheuer, G. (Eds.) *Der Blick nach innen. Aspekte der alltäglichen Lebenswelt Mathematikunterricht. Band 1*. (Münster, Waxmann Verlag) pp. 53-86.

Fetzer, M./Schreiber, C./Krummheuer, G. (2004). An Interactionistic Perspective on Externalisation in Mathematics Classes based on Writing and Graphic Representations. *Proceedings of ICM-10*, CD-ROM.

Fetzer, M. (2003a). Verschriftlichungsprozesse im Mathematikunterricht der Grundschule aus interaktionistischer Sicht, *Journal für Mathematik*-Didaktik, 24 (3/4), pp. 172-189.

Fetzer, M. (2003b). Interaction in Collective Writing Processes in Early Mathematical Learning. *ERME Proceedings 2003*, CD-ROM.

Have ten, P. (1999). *Doing Conversation Analysis* (London, Sage).

Koch, P. & Oesterreicher, W. (1985). Sprache der Nähe - Sprache der Distanz. Mündlichkeit und Schriftlichkeit im Spannungsfeld von Sprachtheorie und Sprachgeschichte, in: Deutschmann, O. & Flasche, H. u. a. (Eds) *Romanistisches Jahrbuch. Band 36* (Berlin, New York, Walter de Gruyter) pp. 1-43.

Krummheuer, G. & Fetzer, M. (2005). *Der Alltag im Mathematikunterricht. Beobachten, Verstehen, Gestalten* (Heidelberg, Berlin, Spektrum Akademischer Verlag).

Krummheuer, G. & Naujok, N. (1999). *Grundlagen und Beispiele Interpretativer Unterrichtsforschung* (Opladen, Leske + Budrich).

Lesser, L. M. (2000). Sum of Songs: Making Mathematics less Monotone! *The Mathematics Teacher, Vol. 93, No. 5,* pp. 372-377.

Morgan, C. (1998). *Writing mathematically. The discourse of investigation* (London, Falmer Press).

Miller, D. L. (1991). Writing to Learn Mathematics, in: *The Mathematics Teacher*, *84 (7),* pp. 516-521.

Miller, M. (1986). *Kollektive Lernprozesse* (Frankfurt a. M., Suhrkamp).

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics* (Reston, VA: NCTM).

Oesterreicher, W. (1997). Types of Orality in Text, in: Bakker, E. & Kahane, A. (Eds.): *Written Voices, Spoken Signs. Tradition, Performance, and the Epic Text* (Cambridge, Massachusetts, London, England, Harvard University Press) pp. 190-214.

Pugalee, K. D. (2005). *Writing to Develop Mathematical Understanding* (Norwood, Massachusetts, Christopher-Gordon Publishers).

Ruf, U. & Gallin, P. (Eds) (2003). *Dialogisches Lernen in Sprache und Mathematik. Band 2: Spuren legen - Spuren lesen: Unterricht mit Kernideen und Reisetagebüchern* (Seelze-Velber, Kallmeyer).

Sacks, Harvey (1996). *Lectures on Conversation* (Cornwall, Blackwell Publishers).

Selter, C. (1994). *Eigenproduktionen im Arithmetikunterricht der Primarstufe. Grundsätzliche Überlegungen und Realisierungen in einem Unterrichtsversuch zum multiplikativen Rechnen im zweiten Schuljahr* (Wiesbaden, Deutscher Universitäts-Verlag).

Toulmin, Stephen E. (1969). *The uses of argument* (Cambridge, University Press).

ISSUES IN ANALYSIS OF INDIVIDUAL DISCOURSE CONCURRENT WITH SOLVING A MATHEMATICAL PROBLEM

Boris Koichu

Technion—Israel Institute of Technology

Individual discourse concurrent with solving a mathematical problem—thinking-aloud discourse—is considered a valuable source of information about thought processes in doing mathematics. Analysis of thinking-aloud discourse is associated with various methodological issues, such as validity or viability estimation and qualitative comparison of thinking-aloud protocols. The paper presents ways of dealing with these issues in an empirical study concerning middle-school students' problem-solving behaviors. Based on examples from that study, we discuss research implications of particular analytical decisions and describe a "safety net" of a multi-step process of drawing conclusions about the students' problem-solving behaviors.

INTRODUCTION

Thinking in silence on a problem is a common situation associated with doing mathematics. *Inner speech* (the term is due to Vygotsky, 1934/1987) produced while solving a problem is of keen interest for several generations of mathematics educators. While direct exploration of one's inner speech is out of reach of the available research techniques, various indirect methods exist and allow the researchers to open a window into the hidden world of thought processes in doing mathematics.

Individual discourse concurrent with solving a mathematical problem—thinking-aloud discourse—is frequently considered a valid source of information about one's inner speech. Validity of a thinking-aloud discourse corresponds to the extent to which subject's talk represents the actual sequence of thoughts mediating solving a problem (Clement, 2000; Ericsson & Simon, 1993). It can partly be tested by subject's reflections on his or her recorded thinking-aloud discourse (Ericsson & Simon, 1993; Schoenfeld, 2002; Van Someren, Barnard & Sandberg, 1994).

Thinking-aloud discourse can be evoked in a proper clinical interview setting. The central assumption of the setting is that it possible to make subjects to vocalize their inner verbal speech in an essentially concurrent manner and even to label some of their non-verbal thoughts by means of pauses, introjections and non-verbal behaviors (Ericsson & Simon, 1993). However, even when the discourse is properly evoked and recorded, its analysis is burdened by powerful limitations and constraints, some of which are unavoidable, whereas others can be reduced or bypassed when special analytical efforts are made (Goldin, 2000; Koichu & Harel, 2007; Van Someren, Barnard & Sandberg, 1994).

This paper is drawn from our experience of making such efforts during a study of middle-school students' heuristic behaviors in solving non-routine mathematical problems (Koichu, 2003; Koichu, Berman & Moore, 2003a; 2003b; 2006a, 2006b).

Two issues, which are broadly discussed (yet not resolved) in the literature about clinical interviewing and thinking-aloud discourse (e.g., Clement, 2000; Ericsson & Simon, 1993; Goldin, 2000), are particularly relevant to this paper: (1) Validity estimation of thinking-aloud discourse; (2) Qualitative comparison of thinking-aloud protocols by different subjects solving one problem (interpersonal comparison).

The goal of the paper is to present analytical decisions made regarding these issues in our research and to discuss their pros and cons—in greater detail than it can usually be done in empirical papers. By pursuing this goal, we hope to address the call for development and explication of methodologies for studying language-based practices in mathematical problem-solving.

ISSUE 1: VALIDITY ESTIMATION OF THINKING-ALOUD DISCOURSE

Consider an example—a 60-sec fragment of thinking-aloud data from Koichu (2003). In this study, 12 middle-school students were individually interviewed for 3 times each at 2.5 month intervals. The students were instructed to think aloud in accordance with Ericsson & Simon's (1993) recommendations. In particular, they were asked not to plan what to say while solving problems or explain their actions to the interviewer. Instead, the interviewees were trained to verbalize their on-going problem-solving thoughts and "act as if you are alone in the room speaking to yourself" (Ericsson & Simon, 1993, p. 378).

Among other interview tasks, the students were given the following one: "Solve the equation $\frac{x-1}{x-1} - \frac{x-2}{x-2} = 0$." Prior to the interview, the students had solved in a classroom many equations with fractions that looked similarly to the given one, but had a finite number of roots. The given equation had an infinite number of roots. The task appeared to be very difficult for the 8th grade students. Most of them were intended to dive into algebraic manipulations trying to reach an equation of the sort "x equals a number". Alon [1], a high-achieving 8th grade student with prominent verbalization skills, was the only interviewee who solved the equation fully and straightforwardly. When given the task, he kept silence for 5 sec and then said:

> Alon: Common denominator *(silence 3 sec)*. No, I don't need even a common denominator! We can see that there are two fractions, first minus second. In the first fraction, the numerator and denominator are equal, and then the fraction equals 1. The second fraction *(silence 2 sec)*, the numerator and denominator are also equal; I can also say that it is 1 *(silence 2 sec, writing "1-1=0")*. Finally, we get 1-1=0, there is an infinite number of solutions *(silence 5 sec, Alon looks at the interviewer)*.

> Int.: So what are the solutions?

> Alon: Solutions of the equation? Well. *(Silence 3 sec).* All the numbers except for 1 and 2, since if we assign 1 or 2 instead of x, we get meaningless expressions.

Let us remark that the transcript seems fairly understandable and enables us to follow some of the students' problem-solving actions. In particular, two analysts independently coded the fragment in terms of a coding scheme focused on the students' heuristic strategies (Koichu, 2003). There is no place here to present the entire process of coding (it is done in detail in Koichu, 2003; Koichu, Berman & Moore, 2003b; 2006a). Briefly, we were looking for markers of the use of the following actions:

> (1) *Planning*, including *Thinking forward*, *Thinking from the end to the beginning* and *Arguing by contradiction*. (2) *Self-evaluating*, including *Local self-evaluating* and *Thinking backward*. (3) *Activating a previous experience*, including *Recalling related problems* and *Recalling related theorems*. (4) *Selecting problem representation*, including *Denoting and labelling* and *Drawing a picture*. (5) *Exploring particular cases*, including *Examining extreme or boundary values* and *Partial induction*. (6) *Introducing an auxiliary element*. (7) *Exploring a particular datum*. (8) *Finding what is easy to find*. (9) *Exploration of symmetry*. (10) *Generalization*.

These strategies are recognized in the literature on mathematical problem-solving (e.g., Schoenfeld, 1985; Larson, 1983) and are described in detail in the aforementioned sources. Regarding the above episode, two coders agreed that Alon started from the act of *planning* based on *activating a previous experience*, which was followed by the act of *local self-evaluating*. The student talked very thoughtfully, and it was difficult to conclude from the videotape whether the student indeed acted as if he was alone in the room or addressed the interviewer.

Six months later we presented the interview videotape and transcript to the interviewee during a three-hour stimulated recall conversation. Alon refreshed his memory of the interview and then explained in great detail how he was thinking on the interview tasks. According to Alon, the most important problem-solving events occurred in and around the short periods of silence. For instance, the utterance "Common denominator…" was the verbalized part of a sequence of thoughts including recalling equations solved in a classroom, procedure of finding a common denominator of two fractions and evaluating its relative difficulty. The decision to cancel fractions and understanding that it would lead to the equation 0=0 came in the 3-sec pause preceding the phrase "I don't need even a common denominator!" At this point, Alon realized that finding a common denominator would lead him to the same equation 0=0 and that the equation is true regardless x. During 2-sec pause after the words "The second fraction", Alon double-checked that the equation 1-1=0 is true regardless x. This thought was reflected in words "this is an infinitive number of solutions" in about 7 sec.

Up to these words, Alon's speech was rather spontaneous, and then he stopped talking. When watching the videotape, the student explained that he was sure that he had completed the solution and was waiting for the interviewer's next move. After 5 sec of silence, the interviewer asked a question "So what are the solutions?" trying to clarify the answer. During a 3-sec pause, Alon interpreted the interviewer's request as

a clue, reconsidered his previous words, recalled that "solution of an equation is a set of numbers that fit it" and understood that he must take into account a domain of the equation. Then he directly addressed the interviewer's question and explained his answer. It seems, also to Alon, that the exact answer "All the numbers except for 1 and 2…" was shaped in the 3-sec pause and the explanation "since if…" was thought of when he verbalized the answer.

We summarize the presented example by the following remarks:

– Thinking-aloud talk is valid in sense that it can reflect the actual sequence of problem-solving steps in an essentially concurrent manner. This finding is consistent with those reported by Ericsson & Simon (1993) in their broad review of research about comparison between subjects who solved problems in thinking-aloud mode or in silence.

– The above reconstruction of the latent part of the transcript, which, in fact, focuses on only one layer of meaning, takes much more words and space than the transcript. There is no one-to-one correspondence between inner and external speech, and even a detailed thinking-aloud protocol of the subject with prominent verbalization skills is incomplete. This is consistent with Vygotsky's (1934/1987) suggestion that inner speech is condensed, abbreviated speech.

– Verbalization of thoughts occurs with shorter or longer delays. As was demonstrated, spoken words reflect thoughts that appear in mind earlier. Metaphorically speaking, two processes – verbalization of one thought and thinking of another one – can take place simultaneously or essentially overlap.

– The interviewee's soliloquy is not free from the social constituent and, in part, is addressed to the interviewer. Moreover, the relationship "interviewer – interviewee" is inevitably asymmetric: the interviewee can interpret as a clue what the interviewer thinks is a "neutral" clarification question (see also Koichu & Harel, 2007). Consequently, the above claim of validity of thinking-aloud discourse holds only for an undistorted talk of the interviewer, prior to any "clarification questions" or "encouraging remarks".

In our research, two methodological decisions were made after the interviews and follow-up meetings with Alon and his friends:

(i) The analysis of concurrent parts of the students' verbal reports in terms of planning-monitoring actions and problem-solving strategies is plausible as it holds both for talks observed during the interviews and for the extended explanations during the follow-up meetings. In essence, the undertaken coding reduces complexity of thinking-aloud data to the level at which possible discrepancies between students' actual problem-solving strategies and their follow-up memories are negligible. Though, some natural limitations of the analysis are related to the analysts' abilities to evaluate the extent to which the

students' thinking-aloud speech is undistorted and to their accuracy in splitting the entire transcript into units of content and assigning the codes to the units.

(ii) It seems impossible to operationally separate strategies *used* and *verbalized* or *self-reported* in problem solving by means of the described research technique. This stimulated us to concentrate on actual heuristic strategies and behaviors of the interviewees rather than on their self-reported ones.

ISSUE 2: INTERPERSONAL COMPARISON OF THINKING-ALOUD PROTOCOLS

In each interview the students were given a warming-up problem (e.g., the above equation served as a warm-up problem of the third interview), a word problem concerning whole numbers (e.g., Problem 2N in Table 1 was used in the 2nd interview) and a geometry open-ended problem concerning quadrilaterals (e.g., Problems 1Q in Table 1 was used in the 1st interview). Thus, 9 different problems were used in 3 interviews with 12 students.

Problem 2N: Represent the number 19 as a difference of the cubes of two positive integers. Find all possible solutions.

Problem 1Q: Check the following statement: If a quadrilateral has two congruent opposite sides and two congruent opposite angles then it is a parallelogram. If you think that the statement is correct, prove it. Otherwise, disprove it by counterexample or by any other method.

Table 1: Examples of the interview problems

The word problems and geometry problems are carefully chosen to meet several conditions. First, technically speaking, the problems could be solved based only on the knowledge available to the middle school students and taught in a classroom not far from dates of the interviews. Second, the interview problems enable both stronger and weaker students to start from the point available for each of them, and then to face a situation of challenge. However, full solutions to the interview problems are out of reach even for high-achieving 8th grade students. Third, the problems are designed to stimulate the students to use a variety of heuristics. Three algebra problems involve opportunities to use all heuristics from the list placed in the previous section; the same holds for the geometry problems. This was evident from piloting the problems with a group of pre-service teachers.

Seventy-two problem-solving episodes ($3 \times 12 \times 2$), excluding warming-up problems, are transcribed and coded in the study. Two trained analysts independently coded the sample protocol (32 content units, 22 minutes of videotape). An agreement rate 84% was found. Thus, we concluded that we could determine with some confidence which heuristics the students used in their solutions. The rest of the protocols were coded by the author of this paper.

Looking at the obtained 72 sequences of the codes, we found them meaningful but overwhelming with respect to the purpose of the analysis—to identify essentially different patterns of the students' heuristic behaviors and estimate to what extent the

patterns were typical. Moreover, we considered devising the patterns directly from the codes unsafe. Indeed, even though systematic mistakes in coding were unlikely because of the reported analytical procedure, occasional mistakes could occur. Practically, we were compelled: to reduce the data to a relatively small number of cases in accordance with some empirical criterion, from those cases to qualitatively describe patterns of heuristic behaviors and then to try to generalize the findings to all the data at hand. In what follows we describe the developed procedure.

Building on the above list of 10 (or 21, including sub-categories) heuristic codes, we operationally defined *heuristic behavior* in solving mathematical problems while thinking out loud: It is a problem-solving behavior, which, from the analyst's viewpoint, is characterized by 5 attributes:

Attribute 1: Number of different heuristics used in the solution of a problem. Two solutions of the same problem with numbers H_1 and H_2 of the different heuristics are considered *similar with respect to Attribute 1* if $|H_1 - H_2| \leq 2$.

Attribute 2: A heuristics used at the beginning of a solution.

Attribute 3: Intention to continue solution in awkward situations with or without asking for assistance.

Attribute 4: When there is more than one attempt to solve a problem, whether or not there is a tendency to use new heuristics that have not been used in the previous attempt(s).

Attribute 5: Typical combination(s) of heuristic steps used in succession.

Then solutions to each problem by different students were compared using the following criteria: Two heuristic behaviors are *essentially similar* if similarity is indicated regarding 4 or 5 attributes and *fairly similar* if similarity is indicated regarding 2 or 3 attributes.

Actually, these criteria are explication of our intention to call some of the interviews similar and the others not. Note that by these criteria two mathematically different solutions can be found heuristically similar. For example, Dalit, solving Problem 1Q, compared the problem's formulation with known theorems about a parallelogram and reached the following conclusion: The quadrilateral is not a parallelogram since the problem's formulation has not been mentioned in the classroom as a theorem about a parallelogram. Kalanit, solving the same problem, "proved" that a quadrilateral is a parallelogram appealing to the "theorem" about congruency of triangles that she called "angle-side-side". These solutions are mathematically different but heuristically *essentially similar* as they are characterized by the attributes 1, 2, 3, and 5. In particular, both solutions are based on *activating a previous experience* with particular reference *to recalling related theorems*.

Mathematically similar solutions in most cases are heuristically *similar* but not necessarily *essentially similar*. For example, Charlie and Hava got the same answer in Problem 2N by checking several positive integers. Their solutions are interpreted

as *fairly similar* since they are similar with respect to the attributes 1, 3 and 4 of the above definition and are not similar with respect to the attributes 2 and 5. In particular, the heuristic search of Charlie included *planning* along *with partial induction* and *thinking backward*, meaning that he planned what examples it is worthwhile to consider, whereas in the heuristic search of Hava *partial induction* was accompanied only by *local self-evaluation*, meaning that a clear plan how to choose numbers for checking had not been articulated.

Similarities among 12 solutions to each interview problem are graphically presented in Figure 1: firm and dotted edges denote essential and fair heuristic similarities, respectively. The students are denoted by the first letter of their pseudonyms in alphabetical order (A=Alon, C=Charlie, D=Dalia, H=Hava, K=Kalanit, etc.). Note that the defined relationship of similarity is not transitive: if solution X is similar to Y, and Y is similar to Z, not necessarily X is similar to Z (see, for example, solutions of Problem 1N by students H, K and L).
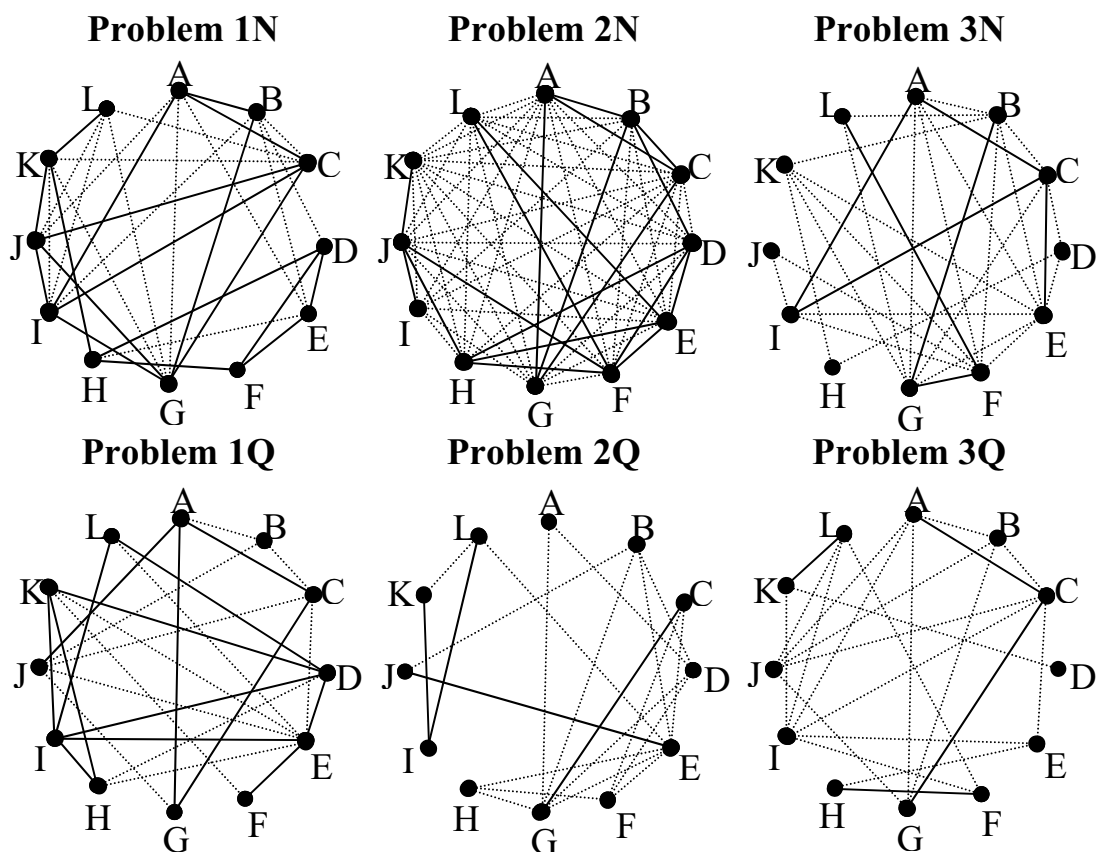

Figure 1: Essential and fair similarities in the 12 students' heuristic behaviours

One to three solutions to each problem were chosen as *prototypical* ones in accordance with the following criterion: (i) the students' speech during the interview was fluent, clear and concurrent; (ii) solutions with maximum similarities to the other solutions were chosen from the solutions that fit the first requirement. Eventually, 72 solutions were reduced to 12 prototypical solutions straightforwardly connected by the relationship of similarity to 54 solutions. Next, five different patterns of heuristics

behaviors—*naïve*, *progressive*, *circular*, *spiral* and *spiral-circular*—were qualitatively deduced from the prototypical solutions. Interested readers can find detailed descriptions of these behaviors in Koichu (2003) and Koichu, Berman & Moore (2006b). There is no space (and need) to present here these findings, and in what follows we continue discussion of the methodological aspects of the study.

We are fully aware that the entire procedure of deducing the patterns of heuristic behaviors, including coding, comparison and reducing the data, heavily relies on a set of empirical (and subjective) analytical decisions. The need to make these decisions manifested itself because of the complexity of the matter investigated. Therefore, it was critically important to estimate *viability* of the research findings. As von Glasterfeld & Steffe (1991) noted, "The most one can hope for is that the model fits whatever observations one has made and, more important, that it remains viable in the face of new observations" (p. 98).

This note was implemented as follows. Obviously, the drawn patterns of heuristic behaviors fit 12 prototypical solutions. We decided to check whether the rest of the data, which were not directly used in devising the patterns, fit the suggested classification. The entire data (72-12=60 solutions) were re-analysed in terms of the deduced patterns. Eventually, we found that the patterns fully fit 50 out of 60 solutions and are applicable to the rest with some qualifications. This finding points to viability of the suggested classification of the students' heuristic behaviors and also to the consistency of the presented multi-step analysis of the students' individual thinking-aloud discourses.

CONCLUDING REMARKS

Analysis of language-based practices in doing mathematics, in general, and of individual discourse concurrent with solving a mathematical problem, in particular, is a complicated yet unreplaceable enterprise in current mathematics education research (e.g., Clement, 2000). What can we learn from students' verbal reports about their *actual* problem-solving behaviors? How can we estimate validity of a verbal report with respect to one's inner speech? What research questions can we reasonably ask? How can we overlook the entire body of overwhelming data? How can we decide which behaviors are more typical than others? How can we estimate viability of the drawn conclusions? This paper presents a particular way of dealing with these general questions and thus addresses the call for making explicit and replicable the current methods of discourse analysis in mathematics education research (e.g., call for papers of WG8 at CERME 5; Duffin & Simpson, 2000; Goldin, 2000).

The described multi-step analytical procedure is comprised of three stages: (1) collecting, transcribing and coding the authentic data; (2) comparison and reducing the coded data; (3) inferring conclusions and viability estimation. Data analysis at every stage included what we think of as a *safety net* of the research—a set of precautions against occasional mistakes, biases and making unsafe conclusions. At the first stage, we considered only undistorted fragments of the interviews, coded the

data in terms that reduce complexity of the verbal reports to the level at which possible discrepancies between students' actual problem-solving strategies and their follow-up memories were negligible and tested reliability of the coding. At the second stage, we designed and implemented explicit criteria for comparison, reducing and sorting the coded data. The criteria helped us "to see the forest" and were robust to possible occasional mistakes in coding. At the third stage, the authentic data were re-analysed for the sake of viability estimation of the inferred conclusions.

In the presented research, the claim about validity of the students' thinking-aloud speech with respect to their inner speech, and, in turn, the claim about correspondence between their *actual* and *verbalized* heuristic behaviors, is supported by two (indirect) means. First, the method of interviewing, thoroughly implemented in the research, is known as one that reasonably assures valid thinking-aloud output (Clement, 2000; Ericsson & Simon, 1993). Second, the analyst-made interpretations of the protocols essentially concurred with the students' follow-up explanations of their performances (Ericsson & Simon, 1993; Schoenfeld, 2002; Van Someren, Barnard & Sandberg, 1994). The drawn heuristic behaviors and the claim that some behaviors are more typical than others are supported by the described multi-step procedure of interpersonal comparison of thinking-aloud protocols.

We tried to show how the developed ways of analysis served in the presented research. Now the natural question arises: To what extent the described analytical procedure may be adapted in other studies dealing with language-based activities in mathematics? Today we believe that it may be helpful, and in time, we will know whether this hope is justified.

ACKNOWLEDGEMENTS

[1] All the names are pseudonymous. The interviews were conducted in Hebrew.

REFERENCES

Clement, J.: 2000, 'Analysis of clinical interviews: Foundation and model viability', in A. E. Kelly and R. Lesh, (eds.), *Handbook of Research Design in Mathematics and Science Education,* Lawrence Erlbaum, New Jersey, pp. 547-589.

Duffin, J. & Simpson, A.: 2000, 'When does a way of working become a methodology?' *Journal of Mathematical Behavior*, 19, 175-188.

Ericsson, K. & Simon, H.: 1993, *Protocol Analysis: Verbal Reports as Data (Revised version)*, MIT Press, Cambridge.

Goldin, G.: 2000, 'A scientific perspective on structures, task-based interviews in mathematics education research', in A. E. Kelly and R. Lesh, (eds.), *Handbook of*

Research Design in Mathematics and Science Education, Lawrence Erlbaum, New Jersey, pp. 517-545.

Koichu, B.: 2003, *Junior high school students' heuristic behaviors in mathematical problem solving*, Unpublished Doctoral Dissertation, Technion, Haifa.

Koichu, B. & Harel, G.: 2007, 'Triadic interaction in clinical task-based interviews with mathematics teachers', *Educational Studies in Mathematics*, 65(3), 349-365.

Koichu, B., Berman, A. & Moore, M.: 2003a. 'Changing teachers' beliefs about students' heuristics in problem solving'. *Electronic proceedings of CERME3.* Retrieved October 9, 2006 from http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG12/TG12_Koichu_cerme3.pdf

Koichu, B., Berman, A. & Moore, M.: 2003b. 'Very able students think aloud: An attempt at heuristic microanalysis', in E. Velikova, (ed.), *Proceedings of the 3rd International Conference "Creativity in Mathematics Education and the Education of Gifted Students"*, University of Rousse, Rousse, pp. 318-325.

Koichu, B., Berman, A. & Moore, M.: 2006a, 'Heuristic literacy development and its relation to mathematical achievements of middle school students', *Instructional Science* (Published online). Retrieved September 25, 2006 from http://www.springerlink.com/content/c248g05416l87556/fulltext.pdf

Koichu, B., Berman, A. & Moore, M.: 2006b, 'Patterns of middle school students' heuristic behaviors in solving seemingly familiar problems', in J. Novotna, H. Moraova, M., Kratka, and N. Stehlikova (eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education, Volume 3*, Charles University, Prague, 457-464.

Larson, L. C.: 1983, *Problem solving through problems*, Springer, New York.

Schoenfeld, A.: 1985, *Mathematical problem solving*, Academic Press, New York.

Schoenfeld, A.: 2002, 'Research methods in (mathematics) education', in L. D. English (ed.), *Handbook of International Research in Mathematics Education*, Lawrence Erlbaum, New Jersey, pp. 435-487.

Van Someren, M. Y., Barnard, Y. F. & Sandberg, J. A. C.: 1994, *The think aloud method: A practical guide to modelling cognitive processes,* Academic Press, London.

Von Glasersfeld, E. & Steffe, L. P: 1991, 'Conceptual models in educational research and practice', *Journal of Education Thought* 25 (2), 91-103.

Vygotsky, L.S.: 1934/1987, *The collected works of L.S. Vygotsky: Volume 1: Problems of general psychology, including the volume Thinking and Speech*, Plenum, New York.

AUTHORITY RELATIONS IN THE ACQUISTION OF THE MATHEMATICAL REGISTER AT HOME AND AT SCHOOL

Tamsin Meaney

University of Otago

Being able to communicate mathematical ideas is an aim of many mathematical curricula around the world. To achieve this, students need to acquire the mathematics register. This paper uses data from research in which a six-year old child's interactions with others, including her teacher, her peers and her family, were recorded. From one day's recording, initial findings are presented of how authority is manifested in these interactions and the effect that this has on opportunities for the child to acquire the mathematics register. It would seem that how interactions occur in the home are more likely to result in the acquisition of the mathematics register.

INTRODUCTION

In many mathematics curricula, being able to use the mathematics register fluently is seen as important (see Australia Education Council, 1991, NZ Ministry of Education, 2006). Based on the ideas of Halliday (1978), the mathematics register is considered to be the set of terms and grammatical constructions that mathematicians believe are the most appropriate ones for communicating their ideas clearly, succinctly and with the maximum amount of meaning to their listeners or readers. However, the mathematics register including an understanding, of when and how it is to be used, must be learnt (Moschkovich, 2003). This paper reports on research that investigated the opportunities both in and outside school for learning the mathematics register. It uses data from the first day of recording to explore how authority within interactions was manifested and the effect this may have had on the child's learning of mathematical language.

The acquisition of the mathematics register occurs over time and has been considered to consist of four stages (Meaney, 2006). First, students have to notice that there is new language to be learnt. With prompting, students use the new terms and expressions. Gradually the prompting is lessen and students begin to use the terms in a variety of situations. Feedback, both positive and negative, helps them to refine their understanding of when and how to use new aspects of the mathematics register. After students have consolidated their understanding, the terms and expressions are integrated into their linguistic repertoire. Students then use these terms consistently, except when the situation is challenging and they may revert back to less matheamtical expressions. The final phase is when students use the terms fluently even in the most demanding situations. Authority for validating the appropriate use of the mathematics register can be expected to reside with the teacher when students are just beginning to learn new aspects of the mathematics register. However, as the students take greater control over when to use it, the teacher's role is one of providing

opportunities for its natural use. The transfer of responsibility in using the mathematics register thus moves from the teacher to the student and can be supported or hindered by the way that opportunities for its use are structured.

Most research about the acquisition of the mathematics register has focussed on what occurs in the classroom (see for example, Chronaki and Christiansen, 2005). Yet, by the time children enter school, they have a significant amount of mathematical knowledge (Young-Loveridge, 1987). By participating in mathematical activities at home, it could be considered that children may also have had opportunities for learning some of the mathematics register. However, this may not be straight forward. Walkerdine's (1988) work suggested that home and school experiences are constituted differently. Although conversations and activities appear superficially to share features so that both can be labelled *mathematical*, the meanings are so different that there is, in fact, little overlap between them. In considering how time is discussed, Walkerdine wrote '[i]n order to operate on the mathematical dimension, the focus has to be taken away from the practical and external relations to the internal relations of the numerical sequence of the measurement of time' (p. 109).

Thus, in comparing how the mathematics register is acquired at home and at school, there is a need to investigate not just whether preference is given to mathematical meaning but how this occurs. This includes looking at who controls the interaction and how they do it. In research about the use of authority to validate what was mathematically appropriate, the teacher was seen as the supreme authority, even though she had verbally tried to dissipate this view (Amit and Fried, 2005). In a study on mathematical explanations, the teacher used her authority to talk over the top of students who did not provide her preferred type of explanation (Forman, McCormick and Donato, 1998). In both studies, it was the teacher who was perceived as having the authority to determine what was valued as mathematics or the mathematics register. Forman et al. (1998) suggested that sociocultural theory, especially that of Bakhtin, proposed that students were socialised into conforming to the forms of academic discourse. They stated that 'students will have to struggle to reconcile their own speaking and thinking with that of the teacher' (p. 316). Consequently, the role of the student could be presumed to be passive. Rogoff (1990), however, showed that a child can have an active role in interactions that result in scaffolding strategies being provided by the adult. Interactions that could lead to students acquiring the mathematics register are affected by two considerations. The first of these is to do with how mathematical meaning is given preference. The second is to do with who controls the interactions and how they do this.

METHOD

This paper uses data from the first day of a case study that investigated a six year old girl's acquistion of the mathematics register both at home and at school. The research child was recorded for one day a week, for twenty weeks, in the second half of 2005. The child's parents are Samoan speakers but English was the primary language

spoken at home. The mother was the research assistant for this study and managed the logistics of recording the child's interactions. From when she woke in the morning until she went to school, the child wore a lapel microphone connected to a digital voice recorder. During her mathematics lesson, she was again recorded and the class discussion captured on another voice recorder connected to a conference microphone. After she was collected from school, the child again wore the voice recorder. Her mother listened to all of the recordings and sent to a transcriber those interactions that appeared to be about mathematical practices. The data for this paper came from the first day of recording.

The transcripts were coded in a number of ways, including identifying the language focus, the language learning stage, and who controlled the situation. In this paper, three pairs of extracts from the classroom and home data are presented.

FINDINGS AND DISCUSSION

The first pair of extracts came from the setting up of mathematical activities. In the classroom, the teacher had written the purpose of the lesson, or the learning intention, on a small blackboard. Extract 2 is from the home and is also about setting up a new board game, *Shrek Operation*. Although the child had been agitating for some time to play it, the decision to do so on that day was the mother's. It is unlikely that the child felt that she was obligated to participate and so the authority for setting up the activity could be considered as shared.

Extract 1

Teacher: What's our learning intention? Let's read it.

Class: I am learning my basic addition facts.

Teacher: If we're doing addition facts, are we adding or taking away? Think about that in your head; turn and tell the person sitting beside you; and stop and listen again. Read our learning intention.

Class: I am learning my basic addition facts.

Teacher: What if I do that? What word have I got?

Class: Add.

Extract 2

Adult: Where's the rules; where's the rule how you're playing?

Child: I don't really know.

Adult: Well, you need the rule book. We need to find it. Oh, I know where it is.

Child: I haven't really set it up yet.

Adult: The object of the game is you have to earn the most money by performing successful operations. Have you got 12 parts?

Child: In how many parts?

Adult: Twelve white bits.

Child: ___ ___ one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve. Done

Teacher: So if we're doing addition, are we adding or taking away?

Class: Add.

Teacher: Adding, ka pai [Mäori word for good]. What's the sign that we use? Everybody make the adding sign up in the air with their magic finger; write it on the carpet in front of you; write it on the back of your hand; write it on your palm; write it on your knee; write it on your other knee; write it. So what are we learning to do?

Adult: See, there are two types of cards [child], look specialist, and a …

Right, there are two types of cards.

Child: Specialist and doctor.

Adult: Right, start with a specialist card and we'll deal them up face up one at a time so that each player gets the right number, like this. Oh, sorry, so we can get the same amount. Ok, can you count how many you've got?

Child: I've actually got one, two, three, four, five, six. Six specialist cards.

Adult: Okay, take any extra cards out of the pile and shuffle the doctor cards, take ___ oh yeah, they have to go over there.

Child: I haven't actually finished setting this up, I need the book to see what they look like 'cause I'm not good at them, yeah.

Analysis

Language focus:	Addition (definition of and connection to the mathematical symbol)	Amounts (12, 2, 6, same)
Language skills:	Reading, listening, speaking and writing	Listening, speaking (reading by the mother)
Authority:	Teacher	Rule book (as can be seen in comments by both the mother, at the beginning, and the child, at the end)
Negotiation of	None with the teacher having	By reading parts of the rule book, the mother controlled the setting

interaction:	complete control	up of the game and so orchestrated preference being given to mathematical meanings ('have you got 12 parts' and 'can you count how many you've got'). The child attempted to circumvent this control by suggesting that if she had the rule book, she could set up the game herself. The authority rested with the rule book but who controlled it was negotiated.
Impact on learning of mathematics register:	The way that lesson was organised suggested that the teacher believed that it was new language for the students. Use of all language skills illustrated the connection between them when students interpreted and used 'addition'. Students were invited to connect new aspects of the mathematics register to what they already knew. However, students gave choral responses, thus limiting their opportunities to manipulate what was to be learnt. The teacher was the authority on what was mathematical language and how it should be learnt.	Although the mother had the authority to give preference to the mathematical meanings when she controlled the rule book, she operated as though the child was a competent user of the mathematics register. This is quite different to the classroom example where the teacher used a set of very restrictive questions to ensure that the students used the language appropriately.

Comparison

In the two extracts, the difference is between the type of learning that is occurring. At school, learning is of *what*, in this case addition - how to say it, how to define it, how to write it. The learning at home was more about *when* to use amounts - what contexts do you use them in, why do you use them, how can you control not using them. Both types of learning are needed in order to be a fluent user of the mathematics register. Our data suggests that the classroom interactions tend to do the first type well but the conditions at home also facilitate the second type of learning.

The next pair of extracts are from the middle of activities. Extract 3 began with the teacher asking a series of addition questions to a small group. The questions were written on the whiteboard. Extract 4 is from playing the game. The child was acting as the banker and the interaction was around the need to give money to her mother.

Extract 3

Teacher: Five plus five.

Students: 10.

Teacher: Four plus four.

Students: 8.

Teacher: What's the special name that we give those? They've got a special name, do you know, A?

A: Double.

Teacher: Double, give a big clap.

Students: (Claps)

Teacher: Wow, double. Who knows another double?

Child: Three plus three plus three.

Teacher: That's triple, because how many numbers did she use?

Child: Three.

Teacher: We're doing doubles, and what do you notice when it's a double? Tell me what you notice about the numbers, to be a double they have to be what?

Child: The same, what the same, and I've got one, 200 plus 200.

Teacher: 400.

Extract 4

Adult: Oh, gosh, yeah, so you have to give me two hundred dollars please.

Child: What, two hundred, what does I have to give you?

Adult: Look on the money, does it say two hundred or what?

Child: I see five hundred and one hundred.

Adult: Well, how much is two hundred from that then.

Child: Oh, so it's donkey.

Adult: One hundred plus, if that's one hundred, I want two hundred, how many do you give me?

Child: Mmm, maybe that.

Adult: How many is that?

Child: Two hundred.

Adult: Good because one hundred plus one hundred is…

Child: Two hundred.

Adult: Yeah.

Child: My turn.

Adult: How much money have you got? Are we a new winner?

Child: I've got only five hundred.

Adult: Oh you're beating me, that's more than two hundred.

Child: Oh

Adult: Five hundred is that much, two hundred is that much. (SHOWING HER) That's all, your turn.

Analysis

Language focus:	Doubles	Amounts and the relationship between them (200, 500, 100)
Language skills:	Reading, listening, speaking	Listening, speaking, reading
Authority:	Teacher	Mother
Negotiation of interaction:	Teacher ensured the focus remained on doubles. However, the children's surprising answers to her more open questions meant that they had the opportunity to explore some of their ideas around this focus. The response of 'three plus three plus three' was clarified by the teacher as not being a valid response. The '200 plus 200' response was surprising because, although it was a double, it was not a basic addition fact.	By following the rules of the game, the mother controlled the interaction. She modified her support after listening to the child's responses. There are parallels with classroom extract in this pair, as the child also gives some surprising responses, 'I've got only five hundred'. The mother, like the teacher, uses these to reinforce the mathematical meaning of 'two hundred'.
Impact on learning of mathematics register:	The students who offered surprising responses possibly gained a better sense of the definition of *what* a double was.	The learning is related to the child's immediate interest in playing the game. As the banker with an interest in winning, she needed to know what 200 meant in relationship to 500.

Comparison

In both extracts, the learning is concentrated on the *what*, with the authority lying with the adults. The teacher's focus on the doubles as part of a lesson on basic addition facts meant that surprising results were not built on but rather channelled back to the main focus. The context of the game means that the mother's focus becomes that of the child. If the child is to play the game then she must learn the relationship between the different amounts of hundred dollars. It is possible to imagine classroom activities that would also encourage children to have a purpose to learn. However, the one-to-one situation at home allows for the negotiation in the interaction is perhaps not as easily obtainable in a busy classroom of 24 students.

The final two excerpts both come from when the child (RC) worked with a partner in the classroom to complete a sheet of basic facts. The students discussed the equation $10 - \square = 7$. Soon after extract 5, the research child requested help from her mother,

who was in the classroom for the research. As the project continued, the mother often filled the role of knowledgeable adult with other students as well.

Extract 5

Partner: I know that top one. I know.

RC: Yeah.

Partner: It's ten but that one there?

RC: Oh, don't ask me. Mm, 10, 10 two, 10 take away two, ten take away two.

Partner: 1, 2, 3, 4, 5, 6,7, oh, it's two.

RC: Take away three.

Partner: ____ and two more is

RC: take away three

Partner: Seven and two were 10.

RC: Three.

Partner: Yeah that's true, it's two, it's two.

RC: five, six, seven

Partner: That is actually two.

RC: I will work it out myself.

Partner: See, it's two.

RC: Partner, I will work it out myself

Extract 6

RC: Mum, you help me a bit, 'cause its real, it's a bit hard.

Adult: ___

RC: You could just help me a little bit, it's just that, that, what does, I just need help, what does those arrows mean, and that word mean?

Adult: Which word?

RC: That word, what does?

Adult: Fewer, oh ok, fewer's like less. Look at this, how many more to make seven, so if you've got seven things?

Partner: What goes in here?

Adult: Hold on a second, Partner, I'll just answer a question for Research Child. So if you've got seven things, how many do I need to make ten?

RC: Those two, two.

Adult: Is that right?

RC: Yep.

Adult: No, look.

RC: Bummer.

Adult: How many more do you need to make ten?

RC   Three.

Adult: Okay, see.

RC: So it's three down here.

Adult: Mmm hmm.

Analysis

Language focus:	Amounts (additions to ten)	Amounts in relationship to more and fewer
Language skills:	Reading, writing, listening and speaking	Reading, listening, speaking
Authority:	Each student but with Partner prevailing	Mother (as perceived by both RC and her partner)
Negotiation of interaction:	The students had to complete a worksheet. Partner tried to work with RC but she was not keen to interact.	RC initiates interaction but the mother changes the focus after seeing an incorrect written answer.
Impact on learning of mathematics register:	With each student not recognising the authority of the other, there was limited interaction. A consequence of this is that it was unlikely that much learning occurred.	Although the RC completed her worksheet, she may not have learnt anything more than that she now had the correct answer ('so it's three down there'). This may have been because the responsibility for directing her own language learning was reduced.

Comparison

The context for both episodes is completing the worksheet on basic addition facts. The RC does not value her partner but rather her mother in providing support to get the 'correct answer'. Her lack of valuing of both of her partner's knowledge but also her own knowledge means that, in these episodes, there is no negotiation of meaning. Mathematical terms make the conversation sound like mathematics but in fact little mathematics is likely to have been learnt and misunderstandings were not recognised.

CONCLUSION AND IMPLICATIONS

This initial investigation of this case study data suggests that how authority about mathematics was perceived by our research child did have impacted on her acquistion of the mathematics register. When she needed to know or do something mathematical, then she contributed more to the interaction. In the home situation, the child seemed to have more opportunities for initiating mathematical interactions. As a consequence, the aspects of the mathematics register that arose may have been more easily acquired because they had immediate relevance for the child. However, when the purpose of the mathematical activity was just to complete a worksheet, it is difficult to know how sustained the learning may have been.

Giving preference to mathematical meaning was not something that was done by the child in these pairs of interactions. Rather this was done by the adult, although in extract 2, the child did try to usurp this preference for the mathematical meaning by gaining control of the rule book that was the default authority in this interaction.

This investigation raises some interesting points about how authority affected the acquisition of the mathematics register. For an activity to support the acquistion of the mathematics register, it would seem that the child should be more actively involved in the learning interactions. This seemed to occur more readily in the home environment. At some point, it may be also that the child needs to recognise her own authority in knowing about mathematics. More work is also needed to identify other ways to acquire the mathematics register so that a student's acquisition of it does not have to result always in ventriloquating that of their teacher or other adult (Forman, McCormick and Donato, 1998).

REFERENCES

Amit, M. and Fried, M.: 2005, 'Authority and authority relations in mathematics education: A view from an 8th grade classroom', *Educational Studies in Mathematics Education,* 58, 14-168.

Australian Education Council: 1991, *A national statement on mathematics for Australian schools*, Curriculum Corporation, Melbourne.

Chronaki, A. and Christiansen, I. M.: 2005, (eds.) *Challenging perspectives on mathematics classroom communication*, Information Age Publishing, Greenwich, CT.

Forman, E. A., McCormick, D. E. and Donato, R.: 1998, 'Learning what counts as a mathematical explanation', *Linguistics and Education,* 9(4), 313-339.

Halliday, M. A. K.: 1978, *The social interpretation of language and meaning*, University Park Press, London, pp. 194-204.

Meaney, T. (2006). Mathematics Register Acquisition, *set* 3, 39-43.

Moschkovich, J.: 2003, 'What counts as mathematical discourse', in N. A. Pateman, B. J. Dougherty and J. Zillox (eds.), *Proceedings of the 27th Annual Meeting of the Group for the Psychology of mathematics education*, University of Hawaii, Honolulu, vol 3, pp. 325-331.

New Zealand Ministry of Education: 2006, *The draft New Zealand Curriculum Framework*, Learning Media, Wellington.

Rogoff, B.: 1990, 'The joint socialisation of development by young children and adults' in V. Lee (ed.), *Children's learning in school*, Hodder & Stoughton, London, pp. 41-58.

Young-Loveridge, J. M.: 1987, *The Development of Children's Number Concepts*, Education Dept., University of Canterbury, Christchurch.

IDEA GENERATION DURING MATHEMATICAL WRITING: HARD WORK OR A PROCESS OF DISCOVERY?

Morten Misfeldt

Learning Lab Denmark, Danish University of Education

This paper considers idea generation during the mathematical writing process. Two contrasting explanations of the creative potential in connection to writing are presented; writing as a process of setting and obtaining rhetorical goals and writing as a process of discovery. These views are then related to two empirically found categories of functions that writing serves researchers in the field of mathematics, concluding that both views contribute to understanding the creative potential in relation to mathematical writing.

MOTIVE

The relation between the psychological origin of mathematical ideas and insights and linguistic and semiotic activities regarding mathematics is a matter of current research in mathematics education. The relation between mathematical writing and mathematical meaning has been investigated in classrooms focusing on the written reports of students (Morgan 1998), and amongst researchers (Burton & Morgan 2001, Misfeldt 2006). In this text I attempt to use two conceptualisations of normal alphabetic writing *processes* to learn about the creative potential in the mathematical writing process. The two conceptualisations of alphabetic writing attempt to explain the creative potentials of writing, and the purpose of this paper is to see what the explanations would mean if we considered mathematical use of written representations as a writing process.

TWO CONCEPTUALIZATIONS OF THE WRITING PROCESS

One of the important questions for theories about writing processes is to what extent one can talk about 'discovery through writing'.

I will present two views on this question. On the one hand is the idea of discovery writing, which states that the actual process of writing generates fundamentally new ideas in an unpredictable manner. The other view is that ideas and arguments are produced mentally without any significant influence of writing. The theories that follow this latter idea do acknowledge that knowledge is generated in the process of writing, but see this as an effect of the writers' attempt to put herself in the place of the reader in order to formulate herself in an understandable manner, rather than a 'spontaneous' interaction with written representations. Hence I have labelled this approach to the writing process 'rhetorically driven problem solving'.

Writing as rhetorically driven problem solving

The idea of considering writing processes as a psychological phenomenon grew out of an attempt to help children write better that began in the United States in the sixties (Norris, 1994). An important idea was to divide the writing process into steps such as planning, writing and rewriting (Rohman & Wlecke, 1964). Following the idea of planning and rewriting, Flower and Haynes (1980) conducted a central empirical study in the research on writing processes. One of their goals was to demystify the discovery potential in writing. They describe discovery as a metaphor for the amount of hard work that goes into generating new meaning while writing.

Their theory is that this process consists of (mentally) setting rhetorical goals and solving rhetorical problems in order to achieve these rhetorical goals. A rhetorical goal is a goal that has to do with obtaining an intended effect on the reader. One should not think of rhetorical goals as restricted to problems of persuading or seducing the reader into taking the authors' personal position on a moral or political issue (even though this endeavour would be a rhetorical goal). What makes a goal as a rhetorical goal is that it concerns thinking about a potential reader.

Flower and Haynes show empirically that the concept of rhetorical goal setting is important in distinguishing good and poor writers. They report that approximately two thirds of the new ideas that are developed by the writers in their investigation are found as a response to a rhetorical problem whereas only one third of the ideas are generated only as a response to the topic itself.

The idea of writing as a rhetorical driven problem-solving process has been developed further by Bereiter and Scardamalia (1987). They define writing as "The composition of written text intended to be read by people not present." Bereiter and Scardamalia (1987) distinguish between two basically different modes of writing: knowledge telling and knowledge transforming, and develop psychological models describing these two modes.

In a writing-process following the knowledge telling model the writer uses his discourse knowledge (knowledge about creating texts, about different genres, etc.) together with already existing content knowledge to create a text. The composition process starts with a mental representation of an assignment ("write an assay on my summer holiday") that splits up in two components, a discursive (write an essay), and a content oriented (my summer holiday). When an initial idea of form and content is established, the writer searches his memory for knowledge, all knowledge is tested for appropriateness and finally written text is generated. Even though the writer's content knowledge and discourse knowledge are both brought into play, these two types of knowledge do not interact. Bereiter and Scardamalia do consider writers that use the written text actively in knowledge production:

"[…] they are used to considering whether the text they have written says what they want it to say and whether they themselves believe what the text says. In the process,

they are likely to consider not only changes in the text but also changes in what they want to say. Thus it is that writing can play a role on the development of their knowledge." (Bereiter & Scardamalia, 1987, p. 11).

To explain such a transformation of knowledge, Berieter and Scardamalia use the 'knowledge transforming' model. Here writing is considered as imbedded in a cognitive problem-solving process where the knowledge telling process plays just one small part. The problem-solving goes on in two different psychological 'spaces', one concerned with content related problems and the other concerned with rhetorical problems, but the two spaces interact; a writer working in the rhetorical problem space with an issue of clarification may end up deciding that she needs to clarify the concept of 'responsibility' and work on this concept in the problem space leading to a new insight that will effect the entire text.

Bereiter and Scardamalia (1987) provide empirically evidence that the writing process can follow each of the two models. All writers occasionally experience a writing process that follows the knowledge-telling process, but the more advanced writers tend to use the knowledge-transforming model more often than novices. Both these models of writing operate with a 'mental representation of the assignment' that is established prior to the writing activity. This mental representation is in the knowledge-transforming model followed by an explicit planning phase and a problem solving phase in order to figure out what to say and how to say it.

Discovery writing

Scardamalia and Bereiter (1987) and also Flower and Haynes (1980) view writing as the process of finding out how to say what you know and mean on a subject. In this process it sometimes happens that you uncover gaps or inconsistencies in your knowledge on a given topic. According to David Galbraith (1992), these theories do not fully explain the experience of generating new knowledge while writing.

Galbraith (1992) develops what he describes as a 'romantic position' to writing; the purpose is to produce a model that explains discovery writing without accommodating it to the models of Scardamalia and Bereiter or Flower and Haynes.

The romantic position considers writing a process of figuring out both what we mean and what we know on a topic. As soon as a sentence is externalized in writing, another sentence may 'pop up' disclaiming or contrasting the first sentence. Viewed in this way the composing process becomes a process of constant negotiation between viewpoints trough rewriting until some balance (or deadline) is reached. The basic claim is that during a writing process, the author continuously generates new knowledge. This knowledge generation is more due to the continuing changing stimuli that the developing text provides, than it is due to setting and archiving rhetorical goals, as claimed by Flower and Haynes (1980), or due to an interplay between rhetorical knowledge and content knowledge, as proposed by Bereiter and Scardamalia (1987).

The romantic position does acknowledge that we remember all sorts of things, but in reality these memories and other knowledge that we posses do not a priori make a coherent story, this story is found or created through negotiation and association, and would not be possible (at least for some writers) without externalizing ideas through writing. Galbraith states his notion of knowledge that underlies this view on writing as:

"For the romantic position, the writer's knowledge is contained in a network of implicit conceptual relationships which only becomes accessible to the writer in the course of articulation." (Galbraith 1992, p. 50).

The writer might have a lot of ideas about the topic she writes about, and these ideas do not always fit into one consistent argument, they may even be contradicting. Also, new ideas are generated in the writing process. The role of articulation on paper is therefore to bring forth the various positions in order to figure out not only what to say and how to say it, but also to develop one's knowledge on the topic. This process fundamentally changes the "mental representation of the assignment", which the theory of Bereiter & Scardamalia operates with. This constant feedback is not represented in either the knowledge telling or the knowledge transforming model, because they both work under the assumption that the writer starts writing after he has a good mental representation of the task, and that this basic task-representation usually is stable throughout the writing process.

EMPIRICAL RESULTS ON WRITING AND MATHEMATICS

The two models of knowledge production distinguished by Galbraith highlight different aspects of the writing process, and it is therefore obvious to ask whether mathematical writing has a tendency to favour one of these models. '

In an empirical study of researchers' use of writing (Misfeldt 2005, 2006) I have distinguished five different functions that writing serve to the working mathematician.

Here I briefly describe a case of one mathematician (R1) writing process. The purpose is to motivate the introduction of the functions.

R1 uses three different media for writing mathematics (two paper-based and one electronic) and he clearly classifies his work according to the medium in use. The media are blank scrap paper (the flipside of old printouts) for handwriting, a lined pad, also for handwriting, and his computer with an email client and LaTeX (a mathematical typesetting program). He uses the three media in different ways.

R1 uses the scrap paper for personal scribblings, and he explained that these papers only make sense to him while he is working on a problem, and that most of them are thrown away almost immediately.

R1 explained that if something from the scrap paper seems to work out, he will take the lined pad and try to write down as many details as possible. When that is done,

the scrap papers are thrown away, but the lined paper with all the details are kept carefully in a system of binders. R1 explained that the purpose of writing it out in detail is twofold: To make the work accessible later, and to check in detail if his ideas are correct.

When R1 thinks he has enough work for a paper, or a part of one, he will write a LaTeX version of his work. He described that version as "much shorter, to fit a paper". The LaTeX version is sent to collaborators for comments and proofreading, but the main reason for making the LaTeX version is to produce a paper.

R1 explained that the notes on lined paper in his binders contain more information than the finished paper. He keeps the notes partly to be able to go back and investigate ideas he never published, and also so that he can always go back and check how he arrived at a given result. R1 imagines that this would be practical if confronted with questions of how he worked out a specific result, both to be able to defend his results and to more easily acknowledge if he made a mistake.

R1 primarily communicates with collaborators by email. He explained that the content is often very close to the content of the notes on lined paper that he keeps in his binders. To be able to express mathematics in an email he will often use LaTeX code in the e-mail. This gives rise to some extra work with moving the content to another medium.

Based on eleven interviews, like the one with R1, I have distinguished five functions that writing serves to the mathematician. The functions are not to be considered mutually exclusive or a full description of what mathematical writing is:

1. Heuristic treatment consists of getting and trying out ideas and seeing connections.

2. Control treatment is a deeper investigation of the heuristic ideas. It can have the form of pure control of a proposition or be a more open-ended investigation (e.g. a calculation to determine x). It is characterised by precision.

3. Information storage is to save information for later access and use.

4. Communication with collaborators. Such communication can have various forms ranging from annotation of an existing text, comments or ideas regarding a collaborative project to suggestions of parts to be included in a paper.

5. Production of a paper, where writing is used to deliver a finished product intended for publication and aimed at a specific audience.

The main discussion here is around the functions of Heuristic and control treatment. The investigation shoved that control treatment was strongly connected to saving of information for at least five out of the eleven respondents (Misfeldt 2005, 2006).

In order to explore the interplay between control treatment and information storage as a either a direct interaction between external representations and thinking or as an interplay between two psychological spaces, I will present a case of private mathematical writing, where a mathematician (R3) uses one kind of notational systems for personal writing and another kind for communicating to others.

Private writing in mathematics

One of the big differences between the two ways of conceptualizing writing processes is evident from the way these standpoints consider private writing. Private writing can be described as writing that is not intended to be seen to by others. Scardamalia and Bereiter considers private writing a borderline phenomenon outside the scope of their work, and they specifically define writing as writing intended to be seen by others. From the perspective of discovery writing, private writing is a typical knowledge producing activity.

Whether or not there is private writing in mathematics is in a sense obvious, working with mathematics people tend to write things, formulas, diagrams or sentences, but what about longer private writing, such as personal archives? This case shows how information storage can be of a very private nature and still play a significant role in knowledge production, including the role of verification (or control treatment).

The respondent R3 is about 50 years old. He described his writing process as consisting of two main parts, the black notebooks and a paper draft. The paper draft is in a sense the final stage for him, when it is finished he gives it to a secretary who types it.

The black notebooks have the purpose of saving ideas, proofs and other types of information for him to access later. The black notebooks are kept in a chronological order. R3 stresses that it is important for him to keep all "the garbage" out of the black notebooks. Therefore, he uses other pieces of paper and his blackboard to support heuristic treatment, and to some extent also control treatment. He uses a computer algebra system to support heuristic treatment and control treatment. He showed me no records of these working papers because he did not keep such records.

When I asked him about how the black notebook and the paper draft differed, he showed me a page in the notebook (Figure 1) and explained that the stair shaped graphs would typically not be included in a research paper, because they were of an informal and personal nature. To R3 however, these graphs were a very useful and precise way to store his ideas.

Figure 1: The notebook, R3 would usually not write these representations in a paper.

R3 explained that the mathematical community has developed a strong tradition of presenting research in a very formal way. This is obviously an advantage, because it insures the validity of the knowledge produced in the field. Nevertheless, this also means that journal articles give only little insight into what R3 describes as "the ideas" behind the work.

The case shows that R3 writes for himself in the black notebook mainly in order to save information, but also to support control treatment. This notebook does not consist of scattered calculations and drawings, it is definitely fair to describe the content as mathematical writing. The writings in R3s black notebooks are private. R3 has no intention to show this writing to others.

One may ask whether the writing that R3 does in his notebook is discovery writing or rhetorically driven problem solving?

The first answer to this would be that the notebook is essentially private. Hence the writing in it is not intended to be viewed by others; therefore it cannot be rhetorically driven. But in order to classify the notebook, it is important to look at the notebook as a part of a long process hopefully leading to publication. So even though the sheets from the notebook are not supposed to be seen by others, it does represent the beginning of a more formal writing process. A very clear indication that the black notebook also represents rhetorical issues is given by R3's comment about keeping all the garbage out of the notebook.

The notebook contains thorough calculations that R3 performs to check his ideas and to save them. For both these purposes it is very important to write down calculations

very accurately. Hence the knowledge generation that happens while working with the black notebook is a result of a commitment to the standards for writing in the mathematical genre (rule following, showing every step of the calculation etc.), and can be viewed as a result of a rhetorical problem solving process. But it is important to be aware that the intended audience is simply R3 himself in some future, therefore it is also understandable that the notebook contains representations of a nature that is not meant to be included in formal reports or research papers, as for example the step-shaped graph shown in Figure 1.

TWO TYPES OF TREATMENTS AND TWO TYPES OF WRITING PROCESSES

Heuristic treatment is a function of writing that has to do with idea generation, and with seeing connections. Control treatment is also concerned with supporting conceptual activities, but has more a flavour of verification than of idea generation. The two conceptualisations of writing suggest two different sources of new ideas in writing processes; the interplay between rhetorical space and content space and the discovery writing potential. One may ask if and how the distinction between heuristic and control treatment can be related to these two conceptualisations of knowledge production during writing.

Heuristic treatment is the use of mathematical representations to challenge and support thinking, in the line of "coming up with and trying out ideas and seeing connections. Heuristic treatment writing is both private and temporary (Misfeldt 2006). The view of this type of writing as rhetorically driven problem solving might therefore not be meaningful. The way the author thinks that his heuristic treatment writing will be looked at by others is simply irrelevant, because he would typically not show it to others. Discovery writing, on the other hand, seems a reasonable conceptualisation of the heuristic treatment. The view that as soon as a sentence is externalized through writing a contrasting (or maybe logically equivalent) sentence will 'pop up' in the authors' mind, would certainly explain why heuristic treatment is an important function in mathematical writing. The details of how the process of discovery writing or heuristic treatment goes about can be very different from case to case, but the constantly changing text can be seen as a factor that generates the environment for changing thoughts.

The empirically generated category of control treatment does to some extent reflect the idea of rhetorical problem solving, the critical working through calculations and argumentation can be seen as an act of setting oneself in the place of a critical reader. It might be exactly here the analogy with writing processes is challenged. Is the logical nature of mathematical argumentation merely a matter of setting oneself in the place of a reader?

The philosophical version of this question is beyond the scope of this paper, but the example of the black notebook provides some empirical answer. In the case we saw

that even though the notebook was not intended to be seen by others the considerations about what to present in the notebook, and how to present it could, to some extent, be interpreted as rhetorical considerations. Everything that goes into the black notebook should be correct, and the proofs should be written out in detail. But since the audience solely consists of the author himself he feels no obligation to follow social codes that don't suit him, as for instance to exclude graphical representations of problems and solutions.

CONCLUSION

In this paper I have asked if the creative potential of writing mathematics lies in hard work, or in a process of discovery, and I have done that by reviewing some of the most typical frameworks for describing writing processes, namely writing as rhetorically driven problem solving and writing as a process of discovery. And looked into the way these frameworks relate to mathematical activity. In a sense the answer is that mathematical writing is both hard work and a process of discovery. At least we have seen that it is a reasonable hypothesis, that the empirically generated categories of heuristic treatment and control treatment, that are part of most mathematical writing processes, follow these two types of writing processes. Heuristic treatment is a matter of generating new ideas and seeing new connections. Discovery writing does precisely attempt to explain how writing can support spontaneous knowledge generation. Control treatment is characterised by precision and strongly dependent on the cultural mathematical code, in the form of rules and notation. The extent to which the empathic ability to put oneself in the place of the reader is important for the control treatment is contested by the example of private mathematical writing. The close connections that were shown empirically to exist between control treatment and communication or information storage could be explained by control treatment being (partly) a rhetorically driven problem-solving process.

Bibliography

Burton, L., & Morgan, C. (2000). Mathematicians Writing. *Journal for Research in Mathematics Education, 31*(4), 429-453.

Elbow, P. (1999). In Defense of Private Writing, Consequeses for Theory and Research. *Written Communication, 16*(2), 139-170.

Galbraith, D. (1992). Conditions for Discovery through Writing. *Instructional Science, 21*, 45-72.

Morgan, C. (1998). *Writing Mathematically: The Discourse of Investigation* (Vol. 9). London: Falmer Press.

Misfeldt, M. (2005). Media in Mathematical Writing. *for the learning of mathematics, 25* (2) July 2005.

Misfeldt, M. (2006). Mathematical Writing. *Dissertation, Danish University of Education*.

Norris, L. (1994). Skriveproces - teori og empiri. In C. Kock (Ed.), *Lær og skriv*. Copenhagen: Undervisningsministeriet.

Rohman, D. G., & Wlecke, O. (1964). *Prewriting: The Construction and Application of Models for Concept formation and Writing*. Washington D.C.: U.S. Department of Health, Education, and Welfare.

Scardamalia, M., & Bereiter, C. (1993). Technologies for Knowledge Building Discourse. *Communications of the ACM, 36*(5), 37-42.

STUDENTS' MATHEMATICAL INTERACTIONS AND TEACHERS' REFLECTIONS ON THEIR OWN INTERVENTIONS

Part 1: Epistemological Analysis of Students' Mathematical Communication

Marcus Nührenbörger & <u>Heinz Steinbring</u>

University of Duisburg-Essen

The research presented in this paper offers a methodological approach to the epistemological analysis of mathematical sign-systems in communication and interaction. The epistemologically based analysis is applied to a teaching episode in a multi-age class (grades 1 & 2). Communication processes of constructing in interaction mathematical knowledge are seen here from a complementary perspective: (1) The construction process that takes place in the institutional frame of the mathematics classroom; (2) The reflection process of mathematics teachers on the videos and transcripts of the teaching episode showing their own teaching. This paper as the first of two papers concentrates on the first perspective.

1. INTRODUCTION: THE RESEARCH PROJECT AND ESSENTIAL RESEARCH PERSPECTIVES

During the last years, teaching for multi-age classes (grades 1 & 2) has been introduced at several German schools. Especially for mathematics teaching, this has provoked discussions about problems and possibilities of joint mathematics learning by children of different ages. The research project ›Mathematics teaching in multi-age learning groups – interactions and interventions‹ analyses questions concerning this issue. On the one hand, the mathematical interactions of children during the joint partner work as well as the interaction patterns with which teachers intervene in the children's mathematical communication are being investigated. On the other hand, it is examined to which extent teachers can be sensitised for dense interaction processes by means of collegial reflections (cf. Scherer et al. 2004; Scherer & Steinbring 2006).

10 teachers from four elementary schools are participating in the research project with their multi-age classes (grades 1 & 2) for two years. All teachers have been introduced to mathematics instruction in multi-age groups (cf. Nührenbörger & Pust 2006). The partner work of two children (of different age) is videographed in five lessons per school year. The children work in pairs on open or structure-analogue tasks, which are supposed to permit an interaction and reflection from different points of view for both of them. After each term, the teachers of each school meet for a collegial reflection, in which videographed episodes out of their own instruction are watched and analysed with the help of corresponding transcripts.

The epistemological analysis of the documents is concentrated on interaction processes of the student-pairs, which proceed in alteration with teacher interventions: Interactions *shortly before*, *while* and *after* the teacher intervenes. This first part analyses a mathematical interaction scene about the construction of decompositions to a

given ›roof number‹. This case study shows how two children jointly negotiate mathematical meanings about signs and how later these negotiation processes are re-interpreted and ›corrected‹ as a result of the teacher's intervention.

The second part will present the feedbacks the participating teachers have expressed during their collegial reflections observing the video sequences and reading the corresponding transcripts from the episode (analysed in this part).

2. THE ROLE OF MATHEMATICAL SIGNS WITHIN INTERACTIVE MATHEMATICAL TEACHING AND LEARNING PROCESSES – THE NUMBER CONCEPT AS A FUNDAMENTAL EXAMPLE

From the very beginning, mathematics teaching and mathematics learning deals with the use and interpretation of mathematical signs, symbols and symbol systems. Using the example of the number concept and its epistemological characteristics, the role of mathematical signs/symbols shall be characterised as far as it becomes important for the following epistemological analysis of mathematical interactions. Initially, one can distinguish two essential functions for mathematical signs/symbols:

»(1) A semiotic function: the role of mathematical signs as ›something which stands for something else‹. (2) An epistemological function: the role of the mathematical sign in the context of the epistemological interpretation of mathematical knowledge.« (Steinbring 2005, p. 21, Steinbring 2006)

A comparison between linguistic and mathematical signs reveals the following concerning the first function. The linguistic sign or word ›school‹ first stands for a concrete school – maybe the school, which the students attend. But with ›school‹, one can also designate a big number of different concrete schools – of the same or of a different type. This relation between the word ›school‹ and many concrete schools also covers the ideal construct of the general concept ›school‹ as a place of institutionalised teaching and learning scientific knowledge – and a concrete school is the realisation of this abstract idea. Furthermore the sign ›school‹ can be written in different forms (cursive, block letters, etc.) or languages (école, Schule, scuola, etc.) without there being a change in the illustrated relation between the linguistic sign and the concrete referents or in the abstract idea.

The mathematical sign ›4‹ stands for the conceptual number ›4‹, and that ultimately is an abstract conceptual idea from the beginning. In order to facilitate and to activate child-accordant mathematical learning and understanding processes, there is a multitude of didactical situations and materials to which the sign ›4‹ could relate.

One example for such a referential relation between the sign ›4‹ and an object, which this signs stands for, could be the use of little coloured chips: ●●●● ⟵——⟶ 4

Insofar, the sign ›4‹ relates to the four chips, but does not designate these as the actual objects (as for example the word ›school‹ designates the concrete school of a student), but ultimately ›stands for something else‹ which is meant by the four coloured chips, namely always the abstract concept of the number ›4‹. Comparable to

the different writings of the word ›school‹, the mathematical sign ›4‹ can be written in other ways and languages: ›4‹, ›IV‹, ›100‹ (in the binary system), etc. or ›vier‹, ›quatre‹, ›quattro‹, etc. on the one hand, this difference to linguistic signs – namely that mathematical signs/symbols ultimately always relate to a universal mathematical conceptual idea and not to ›concrete mathematical numbers‹ (for example different materials) – illustrates the special epistemological character of mathematical signs.

The mediation between signs and structured reference contexts requires a conceptual mediation (Steinbring 2005, p. 22). The conceptual idea of ›natural number‹ is needed for regulating the relation between the signs/symbols and their accompanying reference contexts. What is this conceptual idea of the number concept?

In contrast to an empirical understanding of numbers as representing concrete objects or as names of sets, such a conception is fundamentally criticized from philosophical and epistemological perspectives. Paul Benacerraf (1984) for instance states that numbers can neither be objects nor names for objects. »I therefore argue, … that numbers could not be objects at all; for there is no reason to identify any individual number with any one particular object than with any other (not already known to be a number)« (Benacerraf 1984, pp. 290/1). But if numbers are not objects, what else are they? »To <u>be</u> the number 3 is no more and no less than to be preceded by 2, 1, and possibly 0, and to be followed by 4, 5…… Any object can play the role of 3; that is any object can be the third element in some progression. What is peculiar to 3 is that it defines that role - not being a paradigm of any object which plays it, but by representing the relation that any third member of a progression bears to the rest of the progression« (Benacerraf 1984, p. 291).

3. EPISTEMOLOGICAL ANALYSIS OF AN INTERACTION EPISODE – KLAUS AND SÖNKE FIND DECOMPOSITIONS OF NUMBERS

In the following, the development of a mathematical interaction between two young students (KLAUS the older and sönke the younger boy) will be analysed. Initially, the two boys are working together, then, the teacher joins them for a short phase and intervenes in the mathematical interaction. This first part presents an epistemological analysis for comparing the autonomous mathematical interpretations of the two boys with the mathematical interpretations, which are discussed later in the teacher's intervention.

At the beginning of the lesson, the ›number house 7‹ (in the roof) is investigated with the whole class. The task is to find all possible addition tasks for the number ›7‹, and then to write them down in ›floors‹. The children then are working in pairs of two. On their desks, there are two roofs with the numbers ›8‹ (Fig. 1) and ›14‹ and a number of paper strips as floors. The two children are to find as many floors as possible. They are building the number house in their own chosen sequence and if
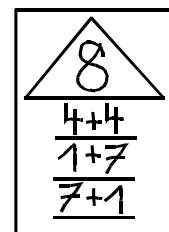

Fig. 1: Number House ›8‹

necessary use chips.

KLAUS and sönke are working together on the number house ›8‹ (Fig. 1). They have already found the decompositions ›4 + 4‹ (written down by KLAUS), ›1 + 7‹ (written down by sönke) and ›7 + 1‹ (written down by KLAUS). In the following, the mathematical interactions of these two boys will be presented in a shortened way.

3.1 Paraphrasing Description and Summarizing Analysis of the Episode

KLAUS and sönke are looking for further additive decompositions to the number ›8‹ in the roof of the number house. Now sönke is to write down the next decomposition.

KLAUS explains the mathematical writing of the number Six. (Phase 1.1 [1-12])

KLAUS asks sönke to write down: six plus two. sönke agrees and writes down on a paper strip: ɔ+2. KLAUS asks which ›letter‹ that (ɔ) is supposed to be; sönke does not understand and KLAUS asks which number it is supposed to be. sönke says: Six. KLAUS adds that the sign (ɔ) looks a little funny; he ›writes‹ down a mathematical 6 on the table with his index finger and remarks that sönke's sign looks like a ›d‹. The according paper strip is taken away and sönke writes down on a new empty paper strip (correctly) 6 + 2.

Analysis aspects: In this short Phase 1.1 the interpretation of the sign ɔ, which sönke has written down, is negotiated. What is the usual convention of writing this number down mathematically? The sign ›d‹, which KLAUS has questioned, is interpreted with relation to different referential features – it could be a letter (a ›d‹) or a number (the number 6). In the epistemological triangle (cf. to this instrument of epistemological interaction analysis Steinbring 2005) this interpretation of the mathematical writing convention for the number Six can be summed up as follows:

With the help of the different referential contexts – offered by KLAUS – for the sign – ɔ – suggested by sönke, the students agree upon the common, conventional mathematical writing of the number Six. The mediation between ›sign/symbol‹ and ›object/reference context‹ is defined *conventionally* and controlled by *conceptual, mathematical* aspects (Steinbring 2005). Thus, in the core it is initially not about a true epistemological – but a conventional – relation between sign and referent.
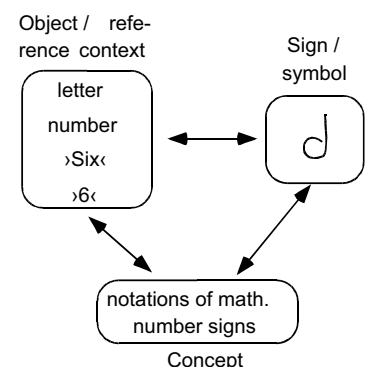

Fig. 2: Epistemological Triangle

sönke finds an exchange task to 4+4. (Phase 1.2 [13 – 36])

KLAUS writes down the decomposition 2 + 6. (Phase 1.2.1 [13])

It is KLAUS's turn and he writes down the decomposition 2 + 6 on a paper strip.

sönke writes down the decomposition 5 + 3. (Phase 1.2.2 [13 – 20])

KLAUS considers which other decomposition is possible and suggests to sönke to write down: five plus three. At the same time, he remarks that there is nothing else left. sönke writes down 5 + 3 on an empty paper strip. While doing this, sönke comments that (in the turn by turn work process) he should ›be at the end‹, and then they are finished. Both, KLAUS and sönke, would have written down the same amount of decompositions.

KLAUS writes down the decomposition 3 + 5. (Phase 1.2.3 [21 – 25])

KLAUS takes the pencil and writes down the decomposition 3 + 5 on an empty paper strip. KLAUS asks whether there is anything else (further decompositions) and he declares that there is nothing else. Meanwhile, sönke is attentively regarding all labelled paper strips and moves them up a little one after the other.

sönke suggests the decomposition 4+4. (Phase 1.2.4 [26 – 36])

sönke opens this sub-phase with the question: »Do you know what I'm simply going to write down there?« (26) and then says again that he has to do once more and then they are finished (with this number house). He then offers his suggestion: »four plus four« (30). KLAUS surprised asks »Again?« (31) and sönke explains that they have every number again, just the other way round. Then, ›four plus four‹ would also be the other way round, one just cannot see it. Following this, he changes the outer writing form of the already noted 4+4 into 4+4 and writes it on an empty paper strip like this. KLAUS is surprised and does not contradict; sönke says that they now have everything.

Analysis aspects: This Phase 1.2 deals with the interactive interpretation of the duplication of 4 + 4. sönke has remarked several times that he has to write down the last decomposition, and that only then is the work on this exercise finished. sönke emphasises the alternating and equal collaboration. Thereupon, sönke introduces and realises the following idea. He wants to write down the decomposition 4 + 4 again. He explains that in the other decompositions each number appears twice, only the other way round (32). Before writing down the same decomposition on an empty paper strip, sönke says »… Four plus four is then also the other way round, just you can't see it.« (34). Thereon, he expresses that these four plus four are written differently and writes the decomposition down like this: 4+4.

In the epistemological triangle (Fig. 3), sönke's interpretation can be presented in this way: sönke explains the repeated decomposition of 4 + 4 with the fact that all decompositions appear a second time. Also, the two decompositions can now be distinguished by means of the new writing. The interpretation of the new ›sign/symbol‹ 4+4 by means of the double appearance of other decompositions and the changed notation are given by sönke rather as a *convention* in this way – and
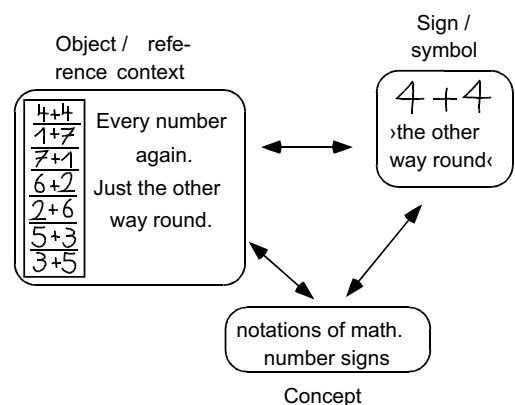
KLAUS accepts it like this. **Fig. 3: Epistemological Triangle**

Implicitly it remains open whether this interpretation is also *mathematically* acceptable and if yes, in which way. The relation between the ›new‹ sign for the number 4 introduced by sönke and the previous decompositions is initially based on the ›surface‹ of the syntactic and visual composition of the sign systems; a more in-depth mathematical-epistemological interpretation of this relation cannot (yet) be observed.

In the following scene, the two boys are working on the number house with the roof number 14. They have already found a number of decompositions of the 14 and written them down on paper strips. When the teacher comes to their table, sönke writes down the decomposition 10 + 4. This is when the episode begins.

The teacher points to the duplication of the task 4 + 4 and missing decompositions. (Phase 2.1 [71 – 92])

sönke writes down the decomposition 10 + 4. (Phase 2.1.1 [71 – 80])

The students are elaborating the decomposition ›10 + 4‹. In this moment, they realise that the teacher is approaching and cast him a look. KLAUS remarks that they have finished their work on the first house. The teacher agrees and waits until the students have finished their current exercise. Meanwhile, he thoroughly examines the paper strips of the first house and moves them apart a little. KLAUS ›dictates‹ sönke the decomposition step by step and accordingly, sönke notes this on a paper strip. They want to switch to writing the next decomposition. Here the teacher intervenes.

Are the two decompositions ›4+4‹ and ›4 +4‹ possible? (Phase 2.1.2 [81 – 92])

Finding out double tasks. (Phase 2.1.2.1 [81 – 83])

The teacher stops the further work on the new number house. He asks the students to check the decompositions in the old number house, and to see whether double tasks appear; these should be taken out. KLAUS points at the upper strip with the decomposition 4+4 and then at the lower one with the decomposition 4+4: »These two«. The teacher confirms this.

›4+4‹ and ›4 +4‹ are no exchange tasks. (Phase 2.1.2.2 [84 – 86])

KLAUS explains that they did it in the same way as with the two decompositions 6 + 2 and 2 + 6 and asks whether one was not allowed to do this with 4 + 4. The teacher assumes that KLAUS thought ›4+4‹ and ›4 +4‹ to be ›exchange addition tasks‹. KLAUS blames it on sönke: »sönke first thought so (*points at S*).« (86)

Decompositions with same numbers in a different sequence. (Phase 2.1.2.3 [87 – 92])

The teacher points at the two paper strips with the decompositions 6 + 2 and 2 + 6 and explains that these are the same numbers, but in a different sequence. Thus, as the teacher explains, these are different tasks. Then he alternately points at the paper strip with 4 +4 and the paper strip with 4+4 and says in comparison to the two previous decompositions that one could not distinguish these two, right? KLAUS agrees and says that he will take them away, which he does immediately. In the con-

versation with KLAUS, the teacher confirms this. At the same time, sönke remarks quietly and by himself – neither the teacher nor KLAUS realise this »But these are different fours.« (91). The teacher closes this phase remarking that in the number house ›8‹, there are still some (decompositions) missing, which should be found,

Analysis aspects: In this Phase 2.1 (especially in the contributions 80 – 92) the teacher interacts with the students and the duplication of the task 4 + 4 is discussed.

Sub-Phase 2.1.2.1. The teacher asks the students to find out double tasks in the number house 8. With the designation ›double tasks‹ a distinction is introduced; there are different and equal – double – tasks on the paper strips. The decompositions are now being marked more distinctively as tasks – which one could calculate for example.

KLAUS immediately reacts to this demand, he points at the two 4 + 4 strips and says: These two. With this, he interprets – as expected by the teacher – the description ›double tasks‹ correctly. In this way, the ›sign/symbol‹ 4+4 or 4+4 receives the mathematical meaning ›equal task‹.

Fig. 4: Epistemological Triangle

With the introduction of the designation ›double *tasks*‹ a mathematical possibility of distinguishing is given: one can now in certain aspects explain mathematically why the decomposition 4 + 4 should not appear twice, as the other tasks.

Sub-Phase 2.1.2.2. KLAUS tries to make clear to the teacher how the two of them have reasoned in order to justify the duplication of 4 + 4. They have referred to the other tasks, eg. 6 + 2 and 2 + 6. But the teacher characterises the tasks 6 + 2 and 2 + 6 as exchange tasks and at the same time denies that the duplication of 4 + 4 is an exchange task as well.

Fig. 5: Epistemological Triangle

After KLAUS has remarked that this has been sönke's idea, the teacher adds a further interpretation. He explains that there are – with 6 + 2 and 2 + 6 – the same numbers, but in a different sequence.

KLAUS and the teacher thus give two different mathematical interpretations of the double decomposition of the 8 into 4 + 4. KLAUS interprets this as illustrated in the epistemological triangle (Fig. 5). The teacher denies this interpretation with the following explanations about what the double decomposition 4 + 4 is *not* as compared to the other decompositions: the new decomposition is not an exchange task and is not a sequence of different numbers, hence a second decomposition of 4 + 4 is not allowed!

Sub-Phase 2.1.2.3. The teacher once again emphasises that one cannot distinguish the two decompositions 4+4 and 4+4 – with the explanations given before.
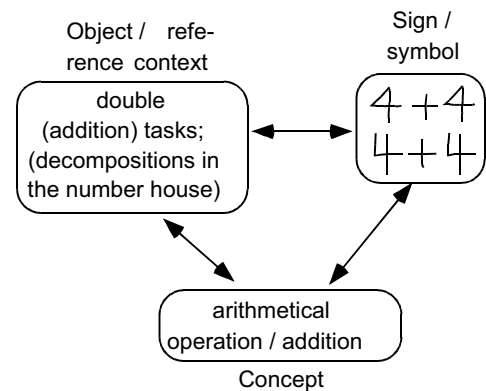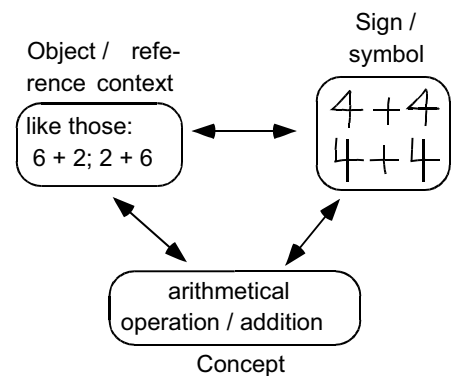
sönke however objects (shyly and quietly): »But these are different fours« (91).

sönke insists – even if quietly and completely unnoticed by KLAUS and the teacher – that these are *different* fours and thus a different decomposition of 8 into 4 + 4.

sönke keeps his explanation of the »difference of the fours«, by means of referring to *conventional* aspects of the notation. For sönke, the fours are distinguishable (different writings), for the teacher, the fours are equal (each time the same task). sönke bases his explanation on writing conventions – not on mathematical relations. The teacher uses the mathematical concepts ›task‹, ›exchange task‹ and ›same numbers in the tasks in a different sequence‹ in his explanation.
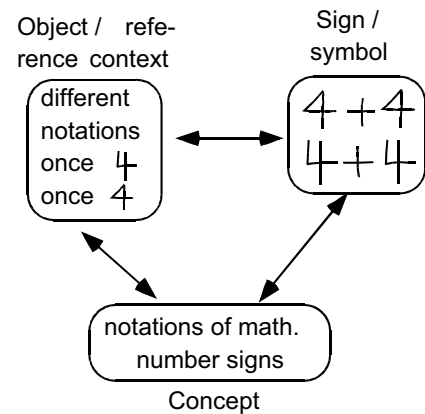
Fig. 6: Epistemological Triangle

3.2 Resume of the Epistemological Analysis

Initially, KLAUS and sönke reach an understanding about how the number six is usually written as a mathematical sign. KLAUS explains that the writing ꓷ is not correct and shows the common form: 6. The referential explanations for the correct mathematical sign for ›six‹ take place on a *conventional* level – no *genuinely mathematical* relations for this explanation are used.

Then, sönke constructs a second decomposition of 8 into 4 + 4. All other decompositions appear twice; this second decomposition is different by means of a new writing: first � and 4+4. This explanation attempts to use mathematical relations. On the one hand, analogies to the other decompositions of the 8 are constructed. The same ›structure‹ for the number of decompositions is requested. In the changed writing of the four, rather *conventional* conditions are brought into play.

Later, the teacher intervenes and discusses reasons why the decomposition 4 + 4 is not allowed to not appear twice: the duplicated decomposition into 4 + 4 is a *double task*, but other decompositions (6 + 2 & 2 + 6) can be seen as two *different* tasks. KLAUS accepts immediately. This explanation is essentially based on the mathematical term ›(addition) task‹ and on the designations ›*different* and *double* or *equal* tasks*‹*, which are thus superfluous. Thus this explanation contains specific mathematical aspects in an elementary form.

KLAUS points to an analogy with other decomposition (6 + 2 & 2 + 6), in order to write down a second decomposition into 4 + 4. This justification is partly based on mathematical ideas, i.e. a consistency with the other decompositions is required.

The teacher underlines that his justification that 4 + 4 may only appear once: The term ›exchange task‹ makes sense only for other decompositions, not for 4 + 4. Furthermore: with exchange tasks, there are the same numbers, but in a different sequence, and thus these tasks are ›different‹. These explanations use elementary mathematical concepts: (addition) task, exchange task, sequence of the numbers

(summands) within the tasks, etc. Of course, there are conventional parts as well, but it is important that with these elementary concepts ›*mathematical* relations‹ between the numbers (in operative connections) are meant or should be meant.

Ultimately sönke insists on his explanation that his different writings of ›4‹ mean that the fours are different and also the two decompositions, a justification, which is founded *purely conventionally* and not specifically *mathematically*.

4. THE STUDENTS' CONSTRUCTION OF MATHEMATICAL KNOWL-EDGE – THE TEACHER'S ›OFFICIAL‹ MATHEMATICAL KNOWLEDGE

The two students KLAUS and sönke have – ultimately in an equal form – interpreted numbers and arithmetical operations. KLAUS has explained to sönke how the mathematical signs of the number six is written. Later, sönke has explained and carried out his suggestion to write down a second – differently written – decomposition of the 8 into 4 + 4; KLAUS has accepted this. The two students have in this way constructed personal and cooperative mathematical knowledge about numbers. With the writings of numbers, like ᗡ (for six) as well as the decompositions �+ᖴ and 4+4, a fundamental epistemological problem (elaborated in section 2) is connected under the outer typographical surface: Mathematical signs/symbols – also with completely different notations – do not relate to ›different concrete objects‹ (learning materials eg.), but are – for example in the epistemological triangle – necessary for the coding of the abstract mathematical concept or the theoretical mathematical knowledge. How can this demanding topic be made understandable for young students at the beginning of elementary school?

In the second interaction phase, the teacher intervenes. Very quickly, a basically changed interaction behaviour between the two students can be observed. The teacher communicates almost exclusively with the older student, KLAUS. The younger one, sönke, can hardly interfere in the discussion. In the common work of the two boys, an equal, collaborative communication and work could be observed initially, which now, with the teacher's intervention, switches to a hierarchic, non-equal communication, strictly focussed on KLAUS. The teacher does not gather information about the discussions and knowledge constructions of the children. He essentially checks the progress and the ›correctness‹ of the previous work. The ›double tasks‹ (the wrong ones) shall be found and removed; additionally, further tasks to the number 8 are to be found before continuing the work on the next number.

The mathematical explanations given by the teacher in order to make understandable why there could not be a second decomposition for 4 + 4, are essentially the following terms: tasks, exchange task, sequence of the numbers (summands) within the tasks. KLAUS accepts these explanations quite spontaneously and takes away the second strip with ›4+4‹; sönke however insists – rather quietly – that there are different numbers and decompositions. Ultimately, the fundamental epistemological problem is not really solved by means of the teacher's intervention. KLAUS accepts that in exchange tasks, same numbers must appear in a different sequence (teacher),

that thus in ›4+4‹ there is no different sequence. He seems to accept the teacher's explanation mainly because of his authority; one cannot discover whether he has gained a far-reaching understanding of the difficult mathematical interpretation problem. For sönke, however, the fours remain different.

The difficult epistemological problem whether there is a second decomposition for 4 + 4 also remains open in the interaction with the teacher and is not really solved. With an empirical interpretation and a visual understanding of the first natural numbers (1, 2, 3, 4, 5, 6, 7, …), that are connected with concrete working material – like the chips used in the lesson here – the children can interpret these numbers as signs to count, put together and separate many concrete numbers. Within this frame, it can be justified that they add ›first the second four chips‹ and ›then the first four chips‹ in the act of calculating and thus reach a different, distinguishable decomposition of 4 + 4. But then, numbers are directly bound to empirical objects. The act of calculating carried out on concrete objects has to be distinguished from the abstract arithmetical operation of addition (cf. Dörfler 2004): in abstract mathematics it is always about the same abstract operation 4 + 4, even if the notation of the signs should be different.

REFERENCES

Benaceraff, P.: 1984, What numbers could not be, In Benacerraf, P. and Putnam, H. (eds), *Philosophy of Mathematics: Selected Readings,* Cambridge, Cambridge University Press, p. 272 – 294.

Dörfler, W.: 2004, Objectifying relations: fractions as symbols for actions, in: Clarke, B., Clarke, D. M. et al. (eds.) *International perspectives on learning and teaching mathematics*. Göteborg: NCM/Göteborg University p. 299-311.

Nührenbörger, M. and Pust, S.: 2006, *Mit Unterschieden rechnen. Lernumgebungen und Materialien im differenzierten Anfangsunterricht Mathematik.* Seelze, Kallmeyer.

Scherer, P., Söbbeke, E. and Steinbring, H.: 2004, *Praxisleitfaden zur kooperativen Reflexion des eigenen Mathematikunterrichts*. Manuscript, Universities Bielefeld and Dortmund.

Scherer, P. and Steinbring, H.: 2006, Noticing Children's Learning Processes – Teachers Jointly Reflect on Their Own Classroom Interaction for Improving Mathematics Teaching. *Journal of Mathematics Teacher Education*, 9(2), 2006, 157–185.

Steinbring, H.: 2005, The Construction of New Mathematical Knowledge in Classroom Interaction – An Epistemological Perspective, Mathematics Education Library, vol. 38, Springer, Berlin, New York.

Steinbring, H.: 2006, What Makes a Sign a *Mathematical* Sign? – An Epistemological Perspective on Mathematical Interaction. *Educational Studies in Mathematics*, vol. 61, 1-2, pp- 133 – 162.

STUDENTS' MATHEMATICAL INTERACTIONS AND TEACHERS' REFLECTIONS ON THEIR OWN INTERVENTIONS

Part 2: Analysis of the Collegial Reflection on Students' Mathematical Communication

Marcus Nührenbörger & Heinz Steinbring

University of Duisburg-Essen

The research presented in this paper offers a methodological approach to the epistemological analysis of mathematical sign-systems in communication and interaction. The epistemologically based analysis is applied to a teaching episode in a multi-age class (grades 1 & 2). Communication processes of constructing in interaction mathematical knowledge are seen here from a complementary perspective: (1) The construction process that takes place in the institutional frame of the mathematics classroom; (2) The reflection process of mathematics teachers on the videos and transcripts of the teaching episode showing their own teaching. This paper as the second part of two papers concentrates on the second perspective.

1. INTRODUCTION: THE ROLE OF COLLEGIAL REFLECTIONS FOR THE PROFESSIONALISATION OF MATHEMATICS TEACHERS

Normally, cooperative working processes of mathematics teachers focus on joint planning and preparing of teaching phases, on joint exchange about "particular" behaviour patterns of children within class and on joint discussions about curricular requirements. Mathematical teaching, however, are rather infrequently introduced by teachers as an explicit object of joint reflection, even if on the one hand the relevance of the development and promotion of reflective abilities on the side of the students is recognised. And on the other hand, the ability to reflect and evaluate one's own teaching, besides the purposeful planning and conduction, is considered an essential characteristic of the professionalism of teachers (cf. Scherer & Steinbring 2006). "The reflections of one's own teaching (…) are a prerequisite for the initiation of children's math learning activities" (Steinbring 2003, 216). Schön (1983) points out that a realisation of one's own activities which is temporally separated from the teaching situation can be initiated with a "reflection-on-action". The "complementary of action and reflection" (Steinbring 2003, 217) leads to the fact that the realisation is connected to future teaching activities, in which the teacher in his action is deliberately related to the results of the distanced reflection ("reflection-in-action", Schön 1983).

During teaching events, a teacher is always directly involved in the interaction with the students and he/she acts according to the implicit behaviour patterns which are estimated as appropriate in the actual situation and are based on routine. He cannot simultaneously take the role of a critical observer and plan an adequate continuation of the action from a distance (cf. Herzmann 2001). Nevertheless, the development of

professional teaching also requires a critical reflection of one's own activities, taken from a distance (cf. Crespo 2006; Scherer & Steinbring 2006). In this respect, the teacher is subject to the tension between immediate involvedness and critical distance. Furthermore, it is essential for mathematics teaching that the mathematical knowledge always newly develops within the teaching interaction with the students, even if the teacher already might possess a consistent and a comprehensive corpus of the mathematical knowledge. "Completely elaborated mathematics thus is no independent input into the teaching process by the teacher, which could then become an acquired output by means of elaboration processes by the students" (Steinbring 2003, p. 196).

A perspective of improving mathematics teaching, in which besides the teaching mathematics is also considered, has been elaborated by Stigler and Hiebert (1999). According to this, the teacher's perception about the special nature of mathematical knowledge (mathematics rather as a product or as a process) determines the way of his teaching activity and the children's learning. The conscious realisation and joint reflection of everyday mathematical teaching activities with "Focus on Teaching, not Teachers" (Stigler & Hiebert 1999) represents on the one hand the essential professional means to improve daily mathematics teaching (cf. Mason 2002). And on the other hand it is meant to help to construct and maintain the "collective memory" of the teaching profession. Steinbring (2003) points out that especially videotaped teaching scenes should be used as the subject of productive reflection meetings by teachers. The video-documentation offers the advantage for the teachers to "consciously distance themselves from situations, to design action alternatives and to evaluate their use in practice" (Selter 1995, p. 116).

The research project "Mathematics teaching in multi-age learning groups – interaction and intervention" examines the productive development of a socio-interactive and self dependent mathematics learning by students in a close interaction with the professionalism of teachers by means of the development of a systematic, joint reflection ability for their own teaching. The first part was about the analysis of interactive constructions by students and teachers within multi-age grouped mathematics teaching. The second part focuses on the distanced, reflective altercations of six teachers with their own teaching activity – here on the same scene "KLAUS and sönke are finding decompositions to the number houses 8 and 14" (an episode from the multi-age group mathematics teaching "1 & 2"). The elementary school teachers participating in the research project – Ben (B), Thea (T), Alwa (Al), Marie (M), Hellen (H), Anja (An) – are following the suggestions made by Scherer et al. (2004) in their cooperative analysis of the teaching episode. On the one hand the discussions focus on everyday language and explication of children in the special context of multi-age learning. On the other hand they as a professional group communicate more general on issues of mathematical and social aspects of teaching. In order to evaluate the documents they use the video scene as well as the respective transcript.

In a first phase, the teachers address the interaction of the two students KLAUS and sönke. Basically, social negotiation processes of children according to their individual roles, which they take because of different points of time of their school enrolment, are almost exclusively in the focus of their attention. Mathematical interaction processes are only addressed marginally. Only the idea of the younger student sönke to construct the exchange task "4 + 4" to "ᛤ + ᛤ", and the following negotiation process between the two students becomes a mathematical topic in the reflection for the teachers. In the following 2nd section will be described and analysed the sequences of the teachers' discussion which is centred on the mathematical-epistemological problem if there exists an exchange task to a task with two equal summands.

In a further sequence, the intervention by the teacher Ben (B) becomes the centre of the collegial reflection. Therefore, in the 3rd section it will be presented to what extent interaction processes of the mathematical students' activities and particularities of the teacher-student-relation are subject to the collegial interpretation.

2. ANALYSIS OF THE COLLEGIAL REFELCTIONS TO THE TEACHING EPISODE "KLAUS & SÖNKE ARE FINDING DECOMPOSITIONS TO THE NUMBER HOUSE 8"

2.1 Paraphrasing Description and Epistemological Analysis

The Teachers Exchange about the Kind of Cautious Prompting (Phase 1.1 [65 – 82])

The interactive efforts of KLAUS to give sönke an interpretation frame for the notation of the cipher "6" are subject of the discussion about the way of helping and thus the relation between prompting and initiating, insisting and encouraging with associative comparisons or images. It is mainly the teacher Alwa who spontaneously expresses rather role-focused and deficit-oriented mathematical expectations to the students' activities and finds this confirmed in the "helper-scene". Alwa addresses the role behaviour of the children, which is typical for the multi-age work in her view. Replying to Alwa's assessment, it is mainly Thea who tries to point out that a mathematical interaction between two children can also be encouraging and productive for both if the older student says something to the younger one. Her expectations on the learning of mathematics of the younger student mainly takes the idea into consideration that mathematical knowledge develops within the interaction and is not transferred from one person to another. Alwa takes into consideration that this subsequent comment does not have to be evidence for the younger student's mathematical knowledge – in this as well as in the following scenes, the teachers often also speak about "the big one" and "the small one". She understands the comment "You can read thoughts" rather as an alibi of sönke who is trying to feign activity.

As a further topic, Marie introduces the way of negotiating the sign ↺. Hellen and Ben interpret KLAUS' comments as evidence for his understanding the sign ↺ noted by sönke as a letter and thus KLAUS inquires. Marie, however, addresses KLAUS' sensible intuition in this helper-scene, in which he explains on the one hand the ne-

cessity of an unambiguous notation of ciphers and on the other hand he cautiously points out the correct notation. Alwa and Thea also perceive the silent, whispering prompting as a continuous principle of KLAUS' helping. Whereby Thea also addresses the way of communication among children who can communicate with few words. By means of the altercation with the objections and the differentiated descriptions of KLAUS' social activities, Alwa realises that he is actively engaged in the work as well and does not only transfer his knowledge.

In this scene, the colleagues mainly negotiate their understanding of social relations between the children. The meaning of mathematical visualisation aids as helping tools or the difference in the situative context of a help – rather product-oriented (as with the notation of ciphers) or rather process-oriented (as with a calculation task) – are barely addressed by the teachers. With a view on the social roles of children, the conversation between the teachers mirrors a development process of the group from rather an initially fixed and unambiguous view on the children's cooperation in a multi-age group to a more open interpretation, which are specifically proven by means of comments in the transcript.

The Reflection about the exchange task to "4 + 4" (Phase 1.2 [101 – 166])

KLAUS and sönke are Developing a Working Structure (Phase 1.2.1 [101 - 118])

The teachers point out the notation of task and exchange task as they are developing within the interaction between the children and their "working structure" (105): At first, they emphasise that KLAUS gives the mathematical pairs, while sönke has only "noticed in the course of the work that there are always two tasks" (109). In the further course of the analysis-conversation the teachers point out the dynamic interactive process, in which KLAUS reflects mathematical connections as the possibility of exchange tasks as well as the systematic arrangement of the single task pairs ("ascending structure", 115).

Is there an exchange task to "╬+╬"? (Phase 1.2.2 [119 - 166])

Alwa points out – again supported by rather general expectations about the understanding of older children – the knowledge of KLAUS that sönke's idea of writing down an exchange task to "4 + 4" is mathematically not correct: He has already understood it, that it cannot be "double" (119). At the same time she admits that KLAUS lets sönke go on. Following this, Ben initiates a discussion about the "crux of the matter" (120) of the task: Is there an exchange task to "4 + 4"? The teachers exchange their own ideas about the empirical abstraction of exchange tasks with the help of the imagination of colours, sequences and tiles. Only Thea points out that the idea of finding an exchange task to "4 + 4" has only developed during the working process, as sönke discovers a structural connection between task pairs and transfers this to the task to which no second task has been written down yet. This suggestion of sönke is interpreted by the colleagues as valuable for KLAUS in order to think more deeply about the connections between task and exchange task.

Analysis aspects: The idea of sönke to create a further completing task, causes different interpretations of the mathematical situation among the teachers: How has the exchange task to "4+4" to be understood? Can a meaningful interpretation be constructed with the help of a reference context?

Ben emphasises the possibility of constructing an exchange task to a decomposition with two equal summands as a "crux of the matter", which touches his mathematical understanding about the view of numbers. He considers that the exchange task with two different summands is visible, whereas one cannot discover it "from the outside" with two equal ones in the level of the signs (120). As an argument for his point of view and the one of sönke, who makes the change of the number sequence "visible", Ben quotes the possibility that both summands could be represented by means of differently coloured chips in spite of the equal number.

In the epistemological triangle (see part 1), this interpretation of the mathematical operation perception can be represented as the followings: The interpretation of the construction of an exchange task to the ›sign/symbol‹ 4+4 is given by Ben on the one side by means of the double appearance of other decompositions, which "have the same result, but are the other way round" (120). On the other side, he constructs his mathematical-epistemological interpretation of the relation between two "exchange tasks" by means of abstracting the signs empirically to differently coloured chips. He distinguishes between the one and the other "4".

Fig. 1: Epistemological Triangle – Ben

In opposition to Ben, Thea addresses an interpretation of the interaction, which takes up and tries to emphasise sönke's point of view (see part 1 of this paper). Insofar, she does not really describe her understanding of the particular relations of tasks and exchange tasks. She sees a reference to a structural mathematical understanding of exchange tasks. The relation between the "new" signs for the number four introduced by sönke and the previous decompositions is thus founded less on the "surface" of syntactical and visual

Fig. 2: Epistemological Triangle – Thea

arrangement of the sign systems but rather on a concession to the observer in order for him to realise that as with the other tasks, also "4 + 4" has been "turned around".
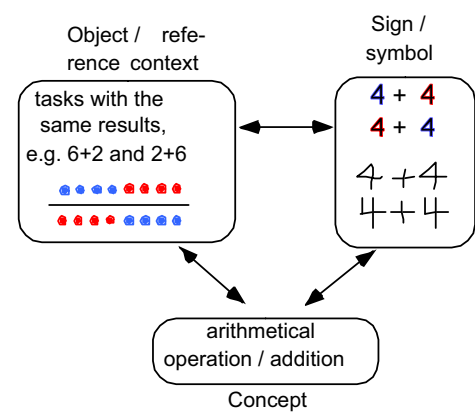
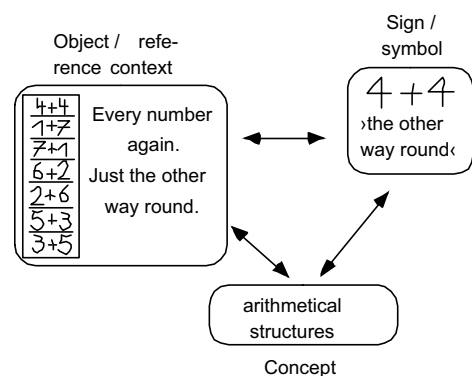Ben's epistemological-mathematical interpretation of exchange tasks leads to an intensive negotiation process amongst the teachers. Marie contradicts Ben with the comment that according to his "theory", every task with two different summands would possesses three further exchange tasks. Marie distinguishes between a concrete level, on which one can for example change colours and an abstract level, on which the numbers remain the same and thus the exchange task disappears.
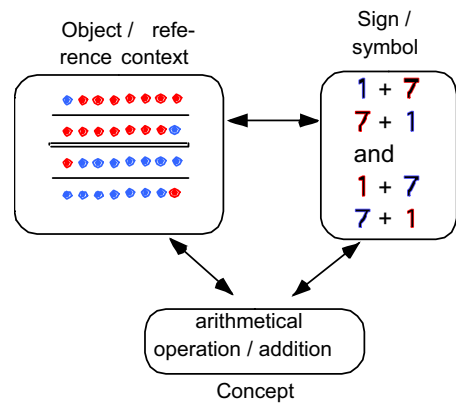
Fig. 3: Epistemological Triangle – Marie

This interpretation of the "disappearance" of exchange tasks on the level of the numbers leads Hellen to refer to similar comments by the children in her class, who talked about the fact that it was "not worth to exchange the numbers with 4 + 4". Ultimately, sönke insists on his explanation that for him, the different way of writing the four means that it is about different fours and thus also about two different decompositions which is a justification to support a *purely conventionally thinking* and not specifically *mathematically* one.

Even if Ben realises the discrepancy within the concrete relations constructed by him, he terminally remains bound to the idea that with exchange tasks on the level of the signs, ultimately the sequence of the numbers is interchanged and thus according to his epistemological understanding two different fours exist. Just like sönke in his interaction with KLAUS, Ben insists on his point of view of numbers. His

justification is founded on conventional aspects and not on specifically mathematical ones. With reference to two different "quantities", which represent only one task independent from their actual sequence, Ben sees also with two equal numbers the possibility of exchanging them. Insofar he continues to understand the construction of task and exchange task as a change of the two summands, independent whether these are equal or different: "There is the set seven and the set one. Whether I have it in the beginning or at the end. And that is the same with the four. Thus with four plus four" (140, 142).

Fig. 4: Epistemological Triangle – Ben

2.2 Resume of the Epistemological Analysis

The analysed collegial reflection about mathematical interaction processes is about an interpretation of mathematical operations, caused by a student's idea. The mathematical problem – what is an exchange task – becomes a topic of negotiation for the
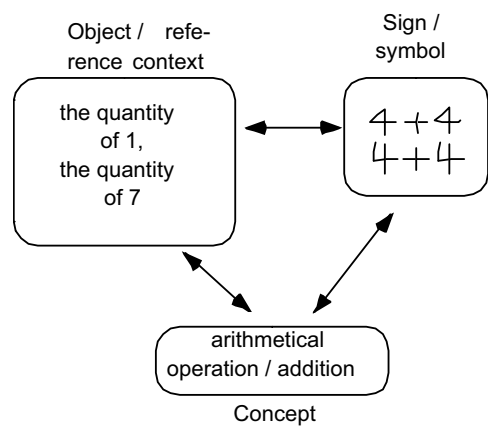
teachers as well as for the students. In contrast to the discussion among the students, the teachers argue on a *conventional* level as well as arguing with a reference to *genuinely mathematical* relations. Ben emphasises conventional conditions as the sequence of the summands as well as concrete materials, which are in a mathematical relation. The given justification is opposed by the other teachers and is pointed out as a wrong conception about exchange tasks within a negotiation process. As opposed to the other teachers, Thea bases her explanation on the "structural relations" between the exchange tasks, which may have been transferred to "4 + 4" by sönke.

The teachers discuss about the question of the exchange tasks but they do not take this as an occasion to talk about concrete constructive ideas in order to dissolve the mathematic-didactical problem of how this topic can be addressed to children within the teaching. Only in the course of Ben's intervention, a teacher aims at this but without receiving concrete mathematics-specific answers by her colleagues. Rather, the advice is restricted to general pedagogical comments about initiating students' contributions and containing themselves.

In the following sequence the teachers' comments about the teaching scene, in which the teacher accompanies the students' working phase, will be analysed in detail. Again, the teachers discuss less the mathematical contents, but mainly the meaning of the teacher's role and the development of the teacher-student-interaction.

3. ANALYSIS OF THE COLLEGIAL REFLECTION ABOUT THE TEACHING EPISODE "THE TEACHER POINTS OUT THE DUPLICATION OF THE TASK "4 + 4" AND MISSING DECOMPOSITIONS"

3.1 Paraphrasing Description and Interpretative Analysis of the Phase 2.2 - The Teachers Exchange about the Kind of Intervention

KLAUS shows Ben the Previous Working Results (Phase 2.2.1 [226 - 245])

The teachers discuss about KLAUS' reaction of interrupting his working process in the presence of the teacher and immediately pointing out the previous working results. They disengage from the transcript and see in this behaviour a typical example of older children, who obviously learn in their academic socialisation that the mathematical learning process is about "getting something done and finished". This result-focusing attitude is communicated by the teacher in the sense of a culture of everyday mathematics teaching. It is mainly Thea who feels a need to stress the discrepancy between the product-orientation and the theoretical intention: "academic learning is not final, but a continuous process, which always goes on" (245).

KLAUS and sönke Change during the Teacher's Intervention (Phase 2.2.2 [253 - 308])

After having regarded the scene again, the teachers realise how much the children change in their working and interaction behaviour when the teacher joins them. This leads to the fact that the observed teacher Ben withdraws from the discussion.

Thea explicitly points out the "controlling" behaviour of Ben, as he immediately starts to sort the task cards to receive an overview. It is discussed that sönke draws back, while KLAUS leads the conversation with Ben, he presents the working results and at the same time refuses to take responsibility for the exchange task to "4 + 4".

The role of the Teacher during the Partner Work (Phase 2.2.3 [309 - 324])

The teachers realise that the partner work between the two students is interrupted: On the one hand, KLAUS does not confer with sönke in his decisions, but rather distances himself from him; on the other hand, Ben leads the conversation almost exclusively with one of the two children. Ben himself reflects on his conversation focused solely on KLAUS, which hinders him to perceive sönke's whispered comment: "The schizophrenic thing is, I as a teacher have given them a partner work, but I do not lead the student-teacher-conversation as a partner-work-conversation" (322).

Analysis aspects: The critical attempts to interpret the consequences of the teacher's intervention bring out different perspectives, which essentially refer to the socio-interactive activities of the students and the teacher:

a) There is a tension between a rather product-oriented interaction behaviour of the older student and the surprising realisation how much this behaviour differs from the process-oriented student-student-interaction. The reflective altercation with the transcript makes the teachers realise a change from the mathematical process to the result, which was not only caused ("I would not have thought that, that the children then still change so much", 266) by the teacher's attendance of the working phase, but also emphasised by way of the intervention ("The working process of the two has been interrupted immediately", 277).

b) There is a tension between a rather mathematical, matter-of-fact exchange about concrete contents between students among each other and a rather rule-oriented interaction between the teacher and students. However, in the presence of the teacher the interaction between KLAUS and sönke about the possible existence of an exchange task to "4 + 4" is shaped less mathematically. They speak about "what is allowed in a mathematic lesson and what is not" (286). The teachers realise the students' uncertainty, which emerges in the intervention with Ben and leads to the fact that KLAUS no longer explains the idea matter-of-factly. The result-oriented, unambiguous point of view onto the students' mathematical activities also avoids an open communication about perspectives onto the "crux of the matter", the exchange task.

c) In the teacher-student-conversation, there exists a tension between the direction towards both students, when rather general social agreements are negotiated, and only towards one of the students – mainly the older one – when mathematical contents are being explained and discussed. His own expectations about the mathematical elaboration process and his own mathematical perspectives towards

a topic lead the teacher to pay less attention to the interaction between the students and not to perceive the working or interaction processes.

The teacher's explanations that the decomposition 4+4 may not appear twice are less a subject of their reflection. The analyses focus on the meaning of the teacher's intervention for the children's interaction behaviour. In the centre is the change from a student-student-interaction with a local process character to a teacher-student-interaction with a local product character directed towards one isolated student.

- It is the older student KLAUS who – without being asked – points out the result of the previous working process, who rather uncritically follows the expectations and given rules of mathematics teaching and who speaks exclusively with the teacher.

- It is also the teacher Ben who brings up the previous working result as a topic, even though this was no longer the subject of the children's mathematical activities, and who directs himself exclusively towards KLAUS and explains to him what is meant by double tasks, without asking for explanations about the students' points of view.

What could one have answered to sönke? (Phase 2.2.4 [325 - 328])

At the end of the scene, Alwa asks for a possible answer to the comment that "₊₊₊₊" and "4 +4" are different fours, which was not perceived by Ben. To her, it is mainly about how one can mathematically explain this "crux of the matter" (120) in a way that sönke would have understood it. The colleagues Thea and Hellen suggest to give this question back to sönke and to ask him for a more detailed justification. With this general-pedagogical advice, the teachers finish their discussion about the meaning of a teacher's intervention to the question of the existence of an exchange task to "4 + 4". Thus, it remains open how one can deal constructively with the students' activities and arguments about the nature of the mathematical number concept.

4. CLOSING REMARKS: SENSIBLE PERCEPTION OF THE STUDENTS' INTERACTIONS AND ONE'S OWN TEACHER INTERVENTIONS IN THE COURSE OF COLLEGIAL REFLECTIONS

The teachers have jointly dealt with a scene from their own teaching. Their discussion alternate from a form of expository talk to an exploratory talk (cf. Crespo 2006). The confrontation with the subject-matter-related interaction process of the students, who have worked cooperatively and collaboratively on a mathematical question, leads on the one hand to a sensible perception of the mathematical interaction between students and of one's own teacher role during the accompaniment of working processes. And on the other hand, teachers are encouraged to exchange fundamental epistemological-mathematical questions by means of the children's spontaneous, unexpected ideas. The altercation with the question of the effects of the teacher intervention leads to the question about the possibilities of maintaining a process-oriented student-student-teacher-interaction and thus to a sensitisation of the teachers for their

own interventions and subject-matter-related interactions between and with students. Within the context of the research project, this development of a "reflection-in-action" (Schön 1983) could have already been proven by the teacher's comments. In view of their involvement in the teaching process and their own implicit patterns of acts and routines, the teachers' reflection provides a confrontation with critical situations of mathematical communication and mathematical behaviour. Furthermore it helps developing professional elements of mathematical teaching and interaction (cf. Mason 2002). Hereby, three aspects mainly seem to be essential besides the organisational frame conditions and the trustful willingness of the teachers to open up for the exchange with their colleagues:

- The collegial reflection is founded on scenes from one's own teaching which are watched on video together and can be analysed with the transcript.

- The practice of instructing is in the focus of this analysis.

- The teachers participating in the reflection work within an innovation context, which is challenged to change one's own action routines.

REFERENCES

Crespo, S.: 2006, Elementary teacher talk in mathematics study group, *Educational Studies in Mathematics*, DOI: 10.1007/s10649-005-9006-0.

Herzmann, P.: 2001, *Professionalisierung und Schulentwicklung. Eine Fallstudie über veränderte Handlungsanforderungen und deren kooperative Bearbeitung.* Opladen: Leske + Budrich.

Mason, J.: 2002, *Researching your own practice. The discipline of noticing.* London: Routledge Famer.

Scherer, P., Söbbeke, E. and Steinbring, H.: 2004, *Praxisleitfaden zur kooperativen Reflexion des eigenen Mathematikunterrichts.* Manuscript, Universities Bielefeld and Dortmund.

Schön, D.A.: 1983, *The Reflective Practitioner. How Professionals Think in Action.* New York: The Free Press.

Selter, C.: 1995, Entwicklung von Bewusstheit – eine zentrale Aufgabe der Grundschullehrerausbildung. In *Journal für Mathematikdidaktik,* 1/2, 115-144.

Scherer, P. and Steinbring, H.: 2006, Noticing children's learning processes – Teachers jointly reflect on their own classroom interaction for improving mathematics teaching. In *Journal of Mathematics Teacher Education*, 9, 157-185.

Steinbring, H.: 2003, Zur Professionalisierung des Mathematiklehrerwissens, In: Baum, M. and Wielpütz, H. (eds.) *Mathematik in der Grundschule. Ein Arbeitsbuch.* Seelze: Kallmeyer, 195-219.

Stigler, J.W. and Hiebert, J.: 1999, *The Teaching Gap.* New York: The Free Press.

CHILREN'S TALK ABOUT MATHEMATICS AND MATHEMATICAL TALK

<u>Päivi Perkkilä & Eila Aarnos</u>

University of Jyväskylä

Chydenius-institute – Kokkola University Consortium

The relationship between mathematics and language is essential and complex. Language makes a connection between real life and formal mathematics. On the other hand mathematics can be seen as symbolic system. According to radical pedagogical view it is impossible for children to learn mathematics without language and connection to real life. In our paper we will highlight children's talk about mathematics and math talk they expressed in our research project. Children's talk is interpreted in semiotic and narrative spirit.

MATHEMATICS AND LANGUAGE

Why is Talk important in Mathematics?

Whether it is written, drawn, gestured, or spoken, the medium of mathematical expression is human language. Mathematics is a specialized language developed to communicate about particular aspects of the world. Mathematical knowledge develops through interactions and conversations between individuals and community. It is an intensely social activity. A major way of participating in a mathematics community is through talk. Children use language to present their ideas to each other, build theories together, share solution strategies, and generate definitions. By talking both to themselves and to others, children form, speak, test, and revise ideas. (Corwin et. al. 1995.) In this paper, concept "talk" means children's inner speech they express in written words and numbers.

Talk about mathematics

Hersh (1986) has answered to the question "What is mathematics?" as follows: "It would be that mathematics deals with ideas. Not pencil marks or chalk marks, not physical triangles or physical sets, but ideas (which may be presented or suggested by physical objects). What are the main properties of mathematical knowledge, as known to all of us from daily experience?

1) *Mathematical objects are invented or created by humans.*
2) *They are created, not arbitrarily, but arise from activity with existing mathematical objects, and from the needs of science and daily life.*
3) *Once created, mathematical objects have properties which are well-determined, which we may have great difficulty in discovering, but which are possessed independently of our knowledge of them." (Hersh, 1986, 22.)*

Malaty (1997, 53) points that there is mathematics in everything that humans have created and in everything that humans have not created. The nature of mathematics

comes up especially then when you try to develop mathematical model from every day situation, and to apply mathematical system for example in the problem situation to another new every day situation (Ahtee & Pehkonen, 2000, 33-34). In school children have to learn formulas, exact proofs, or formalized definitions. Without real life connections this kind of math learning may restrict the talk about math in to formal mathematics.

According to Steinbring (2006, 136) mathematical knowledge cannot be revealed by a mere reading of mathematical signs, symbols, and principles. The signs have to be interpreted, and this interpretation requires experiences and implicit knowledge – one cannot understand these signs without any presuppositions. Such implicit knowledge, as well as attitudes and ways of using mathematical knowledge, are essential within a culture. Therefore, the learning and understanding of mathematics requires a cultural environment.

Mathematical talk: Children making spontaneous expressions and interpretations

According to Worthington & Carruthers (2003, 11) when children make actions, marks, draw, model and play, they make personal meaning. It is the child's own meanings that should be the focus of the developing interest, rather than the child's outcome of an adult's planned piece of work, such as copied writing or representing a person 'correctly'. Like Worthington & Carruthers (2003, 12) we see a child's expression in spirit of Malaguzzi's 'hundred languages', the theme of a poem that refers to diverse ways children can express themselves and that recognizes children's amazing potential in making sense of their experiences: "*The child has a hundred languages, a hundred hands, a hundred thoughts, a hundred ways of thinking, of playing, of speaking. …*"

Worthington & Carruthers (2003, 84) defined five common forms of children's graphical marks: dynamic (marks that are lively and suggestive of action), pictographic (representative marks), iconic (discrete marks of children's own devising), written, and symbolic. These forms can be seen in our data. According to Saarnivaara (1993, 103-104) children interpreting pictures and photos expect of them a resemblance to reality. It is essential that the picture creates a strong feeling of reality in the child. The condition for this is that the work imitates reality faithfully, and is a more or less "perfect" analogy of it. However, it is not only a question of the skillful imitation of reality. The child assumes that the subject matter is also true. We as researchers share this view, and we think that all the children's emotional and mathematical expressions are true.

About semiotics

The Peircean sign-relation consists of "a triple connection of sign, thing signified and cognition produced in the mind." A sign, or representamen, is something which stands to somebody for something in some respect or capacity. It addresses somebody, that is, creates in the mind of that person an equivalent sign, or perhaps a more developed sign. That sign which it creates Peirce calls the interpretant of the first sign. The sign stands for something, its object. It stands for that object not in all respects, but in reference to a sort of idea, which he has sometimes called the ground of the representamen. (Hoopes, 1991; Steinbring 2006, 141.)

The reflection on the interrelationship between sign and meaning is called semiotic activity. Semiotic activity consists in every inter- or intra-personal reflection on the interrelationships between a sign and its meaning(s) in order to investigate and improve mutual correspondence. Signs can be words or graphics (e.g. symbols, drawings, diagrams or schemata). To make a sign and its meaning match optimally, the sign, the meaning, or both can be adjusted: sometimes people modify the sign to make it more adequate for the expression of the meaning, sometimes they elaborate the meaning in order to adjust it to the sign. (van Oers & Wardekker 1999, 234.)

The primary focus in a semiotic perspective is on communicative activity in mathematics utilizing signs. This involves both sign reception and comprehension via listening and reading, and sign production via speaking and writing or sketching. While these two directions of sign communication are conceptually distinct, in practice these types of activity overlap and are mutually shaping in conversations, i.e., semiotic exchanges between persons within a social context. Sign production or utterance is primarily an agentic act and often has a creative aspect. (Ernest 2006, 69.)

Vygotsky (1978) argues that all semiotic functioning is first developed in the young human being through the convergence of several modes of representation, including spoken language, bodily movements associated with drawing and painting, and the use of physical objects as signs, standing for imagined objects in play. Through such modes of expression the power and general properties of the semiotic relation between sign and object, representation and meaning, signifier and signified is first learned and developed.

For example, pictures represent as iconic objects the real world. Children's spontaneous reactions to the pictures are signs, representamen. Children's symbolic relations to mathematics can be interpreted of these signs.

Bruner (1960, 1964) has presented a synthesis of Piaget and Peirce. Bruner claims that root meanings for signs are often constructed by individuals on the basis of enactive, bodily experiences. Subsequently, Bruner argues, these meanings are further developed through internalisations of iconic representations before being fully represented symbolically. While defining the narrative construal of reality, Bruner (1999, 147) use a metaphorical interpretation: "We live in a sea of stories, and like the fish who will be the last to discover water, we have our own difficulties grasping

what it is like to swim in stories. It is not that we lack competence in creating our narrative accounts of reality – far from it. We are, if anything, too expert. Our problem, rather, is achieving consciousness of what we so easily do automatically, an ancient problem of prise de conscience."

According to Doxiadis (2003, 20) mathematical narrative must enter the school curriculum, in both primary and secondary education. The aim is: a) to increase the appeal of the subject, b) to give it a sense of intellectual, historical and social relevance and a place in our culture, c) to give students a better sense of the scope of the field, beyond the necessarily limited technical mathematics that can be taught within the constraints of the school system. In the Finnish National Core Curriculum of Mathematics (2004, 157) is emphasized that children should learn mathematics by talking, modelling, explaining, and presenting their ideas to each other.

The early years of schooling are crucial, as it is often here that the dislike of mathematics is planted. The main cause of this is the difficulty of a young child accepting abstraction and irrelevance – which mostly peaks with the introduction of the concept of number and arithmetic operations. At age five or six, a child lives in a storied internal environment, i.e. an environment cognitively organized by stories of all kinds, of family, of home, of daytime routine, of behaviour, of neighbourhood, of games, of friends, of animals, of dream. The main characteristics of the storied world are integration and emotional richness. With the introduction to mathematics, the child is *de-storied*, a neologism that sounds suspiciously close to "destroyed". We must be very careful when we provide the first bites of the fruit of the Tree of Abstract Knowledge. (Doxiadis 2003, 20.)

One of the basic aims of our research is to help children to create their own spontaneous narratives about mathematics.

AIMS OF RESEARCH

The main aims of this article are:

1. To describe children's talk about mathematics: How children are defining limits of mathematics?

2. To describe children's mathematical talk: What kind of spontaneous mathematical expressions children produce?

DEVELOPMENT OF RESEARCH METHODS

In order to describe and understand meanings of mathematics during childhood, and to study children's emotions towards mathematics and mathematics learning we had to find the way to develop a hermeneutic phenomenological method especially for children aged 6 to 8. The method should be developed also for those children who can not read and write yet. After some common reflections and conversations we started to develop a pictorial test. The basic idea of this test is in the Harter & Pike's

(1984) Pictorial Scale of Perceived Competence and Social Acceptance for Young Children which was presented by Byrne (1996). The other theoretical backgrounds are the theoretical viewpoint of children's spontaneous marks and meaning making (Worthington & Carruthers, 2003), and gestalt psychology (see e.g. Donderi, 2006). For the pictorial test we gathered 37 pictures of mathematical world in a wide sense. The picture sets are:

1. mathematical issues (11) (4 comparisons, 2 one to one correspondence, 5 problems)
2. human beings (7)
3. culture products (7)
4. toys and fairy-tale creatures (6)
5. nature and nature products (3)
6. built environment (3)

The first three sets are most essential in our research. Because the test is developed for children there are pictures about toys and fairy-tale creatures. Children's developing environments consist either nature or built environments or both of them. Picture types are mathematical tasks (9), drawings (12) and photos (16). We have copyright owners' permissions to use their pictures in our test. There are examples of our pictorial test in the next two pictures (1. & 2.) representing the two mathematical worlds.

Picture 1: Bunches of rowanberries   Picture 2: Euro problem

The layout of the pictorial test book is based on gestalt psychology: pictures are bright, scarp and large enough; around the pictures there is enough space for a child to concentrate on one picture at time and to write spontaneously down her/his ideas. The double pages are harmonious considering the content and style. Mathematical issues are surrounded by the real world mathematics.

In order to make the emotional expression easy to children we used a familiar three point's smiley-face Likert-scale (happy, neutral, and sad).

Children were asked to evaluate all pictures from three viewpoints: 1) Is there any kind of mathematics in the picture? 2) How did you felt the mathematics in the picture? 3) Please, write down your own mathematical ideas about the pictures.

DATA GATHERING

We sent our research message via 12 teachers who were studying in our institution. These teachers presented our appeal to both their preschool and the first and second grade colleagues. Twenty volunteer teachers from different parts of Finland announced their and their pupils' willing to take part in our research. So we call this sample as quasi-random. The pictorial test was presented in 23 classes to 299 children from preschool to grade 2. We have got research permissions from children, their parents, teachers, school head masters and chief education officers. Data gathering was organised during the period from January to March 2006. There are the numbers of subjects by grades and by gender in the next two tables (1. & 2.).

Groups	Frequency	%
Preschool	93	31,1
Grade 1	158	52,8
Grade 2	48	16,1
Total	299	100,0

Table 1: Subjects by the grade

Groups	Frequency	%
Girls	154	51,5
Boys	145	48,5
Total	299	100,0

Table 2: Subjects by the gender

DATA ANALYSIS AND RESULTS

The pictorial test was coded as follows:

1) smiley-face Likert-scale: 1 = sad, 2 = neutral, 3 = happy.
2) Children's mathematical expressions (under the pictures): 0 = nothing, 1 = numbers, 2 = exercises (e.g., 2 + 3), 3 = solved exercises (e.g., 2 + 3 = 5), 4 = amount expressions and comparisons, 5 = word problems, 6 = mental models.
3) Children's verbal expressions about mathematics: 0 = no mathematical content, 1 = words, 2 = sentences, (besides these contents we also looked for children's emotions from their verbal expressions: 3 = happy, 4 = sad).

Talk about mathematics

The pictures of pictorial test were grouped into two sets: the traditional school math pictures (18), and the 'everyday' math pictures (19). Children's emotional and verbal opinions of mathematics were described with two scales: school math (formal math), and 'everyday math' (informal math). Then we analysed by medians and quartiles children's positions in math world. Mathematics symbolically meant just school mathematics for a part of the children (ca. 10 %). Some children (ca. 10 %) symbolically were attached to 'everyday' math. We are wondering if they have taken up negative attitude towards school math. Most children had very strict opinions like: "You can not find any math in this picture." or "Oh, this is great! This is mathematics!" Some children were considering the limits of math like: "You can not count anything of this picture because it is music!" As Hersh (1986, 22) has argued

well-determined mathematical objects may effect some difficulties in rediscovering if the connection to daily life activities is broken. Children's early life experiences form their conceptions about mathematics and about themselves as mathematics learners.

In classroom interactions, the learners are to become familiar with different forms of mathematical signs in the interaction to acquire their use by means of social participation, and not to use finished given signs according to strict rules. For the learners the signs and the forms of their interpretation develop by means of mediations to reference objects. Further, a generalizing use of the signs in a certain way develops only gradually in the temporal, interactive development; one cannot give the learner finished mathematical signs in their essential meaning at the beginning of her or his process of learning. (Steinbring, 2006, 145.) We wonder if children's strict conceptions of mathematics are expressions about finished mathematical signs and strict rules. These conceptions could be expanded in the spirit of Steinbring.

Mathematical talk

We grouped children's spontaneous mathematical expressions into three sets: amount expressions and comparisons, word problems, and mental models. Frequencies are presented in table 3.

Math talk	% (n=299)
Amounts	10.7 %
Word problems	22.7 %
Mental models	22.1 %

Table 3: Frequencies of math talk

The traditional model of mathematics learning often is understood as silent counting. Rather small frequencies of math expressions may be a sign of this tradition. On the other hand children had to write down their mathematical remarks and some children may have had troubles in writing.

Some examples of children's spontaneous expressions:

Amount expressions and comparisons:

"You can count berries though it could be rather difficult." (A second-grade boy, Tuomas, about the picture with bunches of rowanberries)

"There is more lemonade in the other bottle." (A second-grade girl, Ida, about the picture of forest)

"There are trees behind the long river." (A second-grade girl, Elisa, about the picture of a bridge scenery)

Word problems:

"How much water you need for the ice sculpture?" (A first-grade girl, Maria, about the picture of an ice sculpture)

"How many yellow papers you find in the picture?" (A first-grade girl, Oona, about the picture of three children and a teacher)

"There are five hundred trees in the forest. A woodcutter comes and fells four hundred and fifty trees. Then there are only fifty trees left." (A second-grade boy, Eero, about the picture of forest)

<u>Mental models:</u>

"You can count the black stripes of the bee." (A first-grade girl, Emilia, about the picture of a bee)

"The berries symbolize the task of dividing them in two equal groups." (A second-grade boy, Wili, about the picture with bunches of rowanberries)

"I really do not know but you still can count: pot + pot + pot + pot + pot = 5 pots." (A second-grade girl, Jenni, about the picture of five honey pots)

"This picture is just like a problem task, and I like them very much." (A second-grade boy, Topias, about the picture of euro problem)

"The cat has plenty of stripes to count." (A pre-school girl, Jessica, about the picture of a cat)

While interpreting the pictures children have created in their minds signs and the corresponding meanings which they express in words or graphics. In the spirit of Bruner children have expressed their narrative construal of mathematical reality. Finnish children's good performance in mathematics and science is well-known. Still we are concerned about those children who expressed only numbers and number exercises. We do not know if these children have living senses of numbers and number exercises as Bussi and Bazzini (2003, 216-217) have written about learning algebra.

CONCLUSIONS

Our starting point of this research is to highlight the meaning of real life experiences and signs and meanings most children learn in their early years. According to Presmeg (1998) there is strong evidence that traditional mathematics teaching does not facilitate a view of mathematics that encourages students to see the potential of mathematics outside the classroom. Although some reports indicate that children are involved in many life activities with mathematical aspects, they continue to see mathematics as an isolated subject without much relevance to their lives.

From our point of view, our pictorial test may be seen as a method for children to find mediations between signs and reference objects by means of examples as Steinbring (2006, 141, 157) has argued. The pictures of our test are examples of cultural

environment which Steinbring (2006, 136) sees as a basic requirement for learning and understanding of mathematics.

On the basis of children's mathematical talk presented in our results we can conclude that children have a need to express in words their mathematical ideas and interpretations. Children should have much more opportunities for these expressions – even before they learn to read and write. For example, one preschool teacher wrote down her group's authentic expressions during our data gathering.

We wonder if real life narratives could form an ideal basis for early math learning in school. We wonder if in mathematics learning environments we should respect children's life orientations and action contexts. We want to conclude with the words of Doxiadis (2003, 20): "Save time for narrative, use it to embed mathematics in the soul."

REFERENCES

Ahtee, M. & Pehkonen, E.: 2000, *Introduction to the didactics of mathematical subjects,* (Publ. in Finnish), Edita, Helsinki.

Bussi, M. G. & Bazzini, L.: 2003, Research, practice and theory in didactics of mathematics: towards dialogue between different fields*, Educational Studies in Mathematics* 54, 203-223.

Bruner, J.: 1960, *The Process of Education*, Cambridge, Massachusetts, Harvard University Press.

Bruner, J.: 1964, *Towards a Theory of Instruction*, Cambridge, Massachusetts, Harvard University Press.

Bruner, J.: 1999, *The culture of education,* Cambridge, Massachusetts, Harvard University Press.

Byrne, B.M.: 1996, *Measuring Self-Concept Across the Life Span,* US Washington, American Psychological Association.

Corwin, R. B. et al.: 1995, Supporting Mathematical Talk in Classrooms. *Hands On!* 18 (1).

http://www2.terc.edu/handsonIssues/spring_95/suptalk.html

Donderi, D. C.: 2006, Visual Complexity: A Review, *Psychological Bulletin*, 132 (1), 73-97.

Doxiadis, A.: 2003, Embedding mathematics in the soul: narrative as a force in mathematics education, *Opening address to the Third Mediterranean Conference of Mathematics Education*, Athens, January 3, 2003.

http://www.apostolosdoxiadis.com/files/essays/embeddingmath.pdf

Ernest, P.: 2006, A semiotic perspective of mathematical activity the case of number, *Educational Studies in Mathematics* 61, 67-101.

Finnish National Core Curriculum 2004, The Finnish National Board of Education, Helsinki.

Harter, S. & Pike, R.: 1984, The Pictorial Perceived Competence Scale for Young Children, *Child Development*, 55, 1969-1982.

Hersh, R.: 1986, Some proposals for reviving the philosophy of mathematics, in T. Tymoczko, (ed.) *New directions in the philosophy of mathematics,* Birkhäuser, Boston, pp. 9-28.

Hoopes, J. (ed.): 1991, *Peirce on signs: Writings on semiotic,* University of North Carolina Press, Chapel Hill.

Malaty, G.: 1997, A child on the way to the world of mathematics, in M. Siniharju, M. (ed.) *New directions of the pre-school and the elementary instruction,* Publ. in Finnish, The Government of Education, Helsinki, pp. 51-90.

van Oers, B. & Wardekker, W.: 1999, On becoming an authentic learner: semiotic activity in the early grades, *Journal of Curriculum Studies*, 31 (2), 229-249.

Presmeg, N. C.: 1998, A semiotic Analysis of Students' Own Cultural mathematics, *Paper presented at the Annual Meeting of the International Group for the Psychology of Mathematics Education (PME),* Stellenbosch, South Africa, July, 12-17, 1998.

Saarnivaara, M.: 1993, *The child and art interpretation,* (Publ. in Finnish), Institute for Educational Research, Jyväskylä.

Steinbring, H.: 2006, What makes a sign a mathematical sign? – An epistemological perspective on mathematical interaction, *Educational Studies in mathematics*, 61 (1-2), 133-162.

Vygotsky, L.: 1978, *Mind in Society,* Harvard University Press, Cambridge, Massachusetts.

Worthington, M. & Carruthers, E.: 2003, *Children's Mathematics: Making Marks, Making Meaning,* Sage, Thousand Oaks, CA.

THE INFLUENCE OF LEARNERS' LIMITED LANGUAGE PROFICIENCY ON COMMUNICATION OBSTACLES IN BILINGUAL TEACHING/LEARNING OF MATHEMATICS

Jana Petrová, Jarmila Novotná

Charles University in Prague, Faculty of Education, Czech Republic

The paper is a part of a longitudinal study focusing on qualitative aspects of teaching/learning mathematics in a foreign language. Its aim is to contribute to understanding of the relationship between language and learning. The focus of this contribution is on learners' language and the interferences limited language proficiency can bring to learners' receptive and productive domains. The interferences are analysed, classified and illustrated. Possibilities to eliminate their negative influence on learning are presented. 10-11-year old children in the U.K., Germany and the Czech Republic participated in the study. Neither the teacher nor the children were native speakers of the language of instruction. The aim of the study is to increase teachers' sensitivity to the interferences, their nature and consequences.

1. INTRODUCTION

Bilingual education is a general expression used to refer to any teaching of a non-language subject through the medium of a second or foreign language. In our research, it refers to the teaching and learning of mathematics through a foreign language without any use of the mother tongue. All bilingual programmes including Content and Language Integrated Learning (CLIL) follow the same aim suggesting equilibrium between content and language learning.

Both the subject matter and the foreign language are developed simultaneously and gradually. This is why it is sometimes called *dual-focussed education*. In CLIL classes, the language acquisition naturally goes hand in hand with cognitive development, the integration of content and second language instruction provides substantive basis and exposure for language learning; the language is acquired most effectively when it is learned for communication in meaningful and significant social situations.

In the learning process, a wide range of cognitive processes is activated. Normally, this occurs in the mother tongue. In CLIL, however, mathematical understanding and thinking manifested by the language of mathematics are developed through a foreign language, and conversely, the foreign language is developed through the non-language content. CLIL provides plenty of opportunities for incidental language learning which has been proved to be very effective, deep and long-lasting, (Pavesi et al., 2001). Here, the learners' attention is focused on the mathematical content and thus Second Language Acquisition (SLA) can become non-conscious.

2. RESEARCH QUESTIONS

The teacher's task is to enable the students to develop their idiosyncratic process of knowledge building and meaning construction as well as positive attitudes (DeCorte, 2000). It is a common belief that mathematics and languages are difficult subjects. Therefore, in order to help the learners succeed, it is of the utmost importance for the teacher to examine and analyse possible barriers that might have a negative impact on learning. The CLIL teacher should be able to suggest ways to minimise these and use a variety of effective teaching strategies that would help overcome individual learning difficulties.

The aim of our longitudinal research is to analyze interferences in the interaction during CLIL lessons. We do not attempt to solve general questions of the communication between the teacher and the students; we focus on the following questions:

- Which types of misunderstandings are most connected with the use of the second language?
- How can we identify their causes?
- What can the teacher do to eliminate the negative influence of these interferences?

In this paper, we focus mainly on students' language perception and production.

3. THEORETICAL BACKGROUND

Research on CLIL in mathematics education should address the relationship between language and mathematics learning from a theoretical perspective that combines current perspectives of mathematics learning and classroom discourse with current perspectives on language, second language acquisition, and bilingual learners. Teaching and research are framed by theories of learning in general, theories of mathematics learning and, in this context, theories of SLA. In accordance with Hofmannová, Novotná, Moschkovich (2004) we believe that "theories and empirical results from linguistics, cognitive psychology, and sociolinguistics have laid the groundwork for the study of mathematics learning as it occurs in the context of learning an additional language".

Language use makes the thought processes easier, it also has impact on perception and memory, and it facilitates mental manipulations and representations. With regard to perception of mathematics in monolingual classrooms, possible problems are intralingual whereas in learners who are functionally bilingual the difficulties are of the interlingual nature. Therefore, it is clear that the study of bilingualism is of great importance. (Novotná, Hofmannová, 2003) According to the language of instruction, CLIL classrooms can be organized in different ways: (1) the teacher is a native speaker of the language of instruction and the learners' mother tongue is different from this language, (2) both the teacher and learners are not native speakers of language of instruction. In our research we focus on the situation (2).

Learning mathematics as a discursive activity is described by (Forman, 1996). Theoretically speaking, classroom communication seems to be a simple process covering both receptive and productive skills of learners. In practice, however, the teacher and the class have to make a number of decisions in order to understand the content matter and to make themselves understood successfully. The task is not easy even in the learners' mother tongue. In a foreign language which the learners have been learning for several years only, it is even more complicated. (Hofmannová, Novotná, 2005).

There are diverse approaches to investigating language classroom, the better-known being interaction analysis, discourse analysis, conversation analysis, and variable approaches. (Walsh, 2006) From the above listed approaches, interaction analysis is most closely related to our analysis, as it does not deal only with linguistic aspects of conversation; it includes observation and/or coding instruments. However, majority of its studies do not focus on the special features of CLIL settings. The sets of keys and coding variables in various types of interaction analysis were developed for foreign language classrooms and they need to be modified for the use in CLIL classrooms. Our research attempts to contribute to this movement.

We are studying interferences that are caused by limited language abilities of participants. Under *interference* we understand any disturbance in communication. It may be for example a misunderstanding or an obstacle.[1]   Figure 1

The sources of interferences in CLIL add two more sources in comparison with monolingual settings. To the interferences between the language of mathematics and mother tongue, these between English and mother tongue and English and language of Mathematics (see Fig. 1).
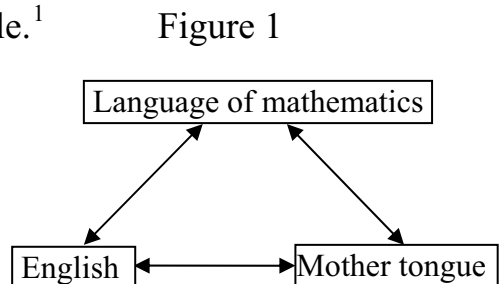
In (Petrová, 2005), the following obstacles that had appeared were presented and characteristics of each of them illustrated by extracts from protocols was given. They were divided into two groups: language problems (unknown terminology, language misunderstanding and mathematical misunderstandings) and communicational specifics of CLIL (ineptitude in speech and limited learners' language production).

4. OUR RESEARCH

Experiments leading to answering of the research questions were carried out in three countries - Great Britain, Germany and the Czech Republic. The aim was to teach the same topic under different conditions – different language competencies both of the learners and of the teacher – and to compare communication under these conditions. In the analysis of all practical experiments, we were looking more deeply at the influence of the language of instruction on the comprehension of the mathematical ideas as well as on the communication in the whole episode.

Experiment design

The *topic* of the teaching sequence was probability. It was presented as a problem of deciding whether a game is fair or not. The mathematical problem dealt with was a non-standard problem. The participating students had not solved similar ones before. This problem was chosen for the following reasons: (a) In order to eliminate the influence of previous experience, a common mathematical topic which the children would already be familiar with was not picked up. (b) It required calculating but not difficult mathematical operations. (c) The problem gave much opportunity to speak. (d) The needed vocabulary did not involve too many special terms and it could be visualised easily. (f) Playing the game and asking if it was fair or not was very motivating and at the same time it was based on real-life experience and investigating.

The approximately 60 minutes long teaching episode was based on playing the game[2], investigation, making graphs, calculating and answering questions, trying to change the game to make it fair and then playing another game in order to verify the new knowledge.

Participants

In each episode, the teacher worked with *groups of two or four children* aged 10-11. The reason for the limited number of participating children was our experience from the pre-experiment clearly showing that work with more numerous groups made the analysis of children's responses and the dual focus (on mathematics content and communication about it) extremely difficult. The whole session was recorded on a tape. The children were usually mixed both in mathematical abilities and in gender. At first we only worked with children whose mother language was the language of instruction (English or German) and later we worked with children who were learning in a second language. Their ability in the second language varied greatly from almost balanced bilinguals to those with a very low ability. For the teacher, it was always bilingual teaching.

The role of the experimenter and that of the teacher were merged in the person of one of authors. We see the main advantage of this organization in the direct access to the sources of the teacher's reactions and decisions in managing the teaching episode. The roles of the teacher and the teacher/observer are different; see e.g. Brousseau (2002) in (Novotná et al., 2003). We believe that merging the role of the teacher and researcher offers to us better understanding of the development of the teaching episodes.

Methods for the collection and analysis of the results

The experiments were audio-recorded. These recordings were later transcribed and analysed. At the same time the teacher's reactions and teaching strategies, and the children's reactions, answers and reasoning were closely looked at. The amount of speech of the teacher and the children were compared. The language was studied

linguistically, the mistakes and communicational obstacles were collected, classified and analysed.

In our ad hoc approach, the interactional features enabling the analysis of teacher's and pupils' interactions were defined. The dual-focus (mathematics and English as the language of instruction) was preserved. The analysis was done from the perspective of production and reception.

4. RESULTS

No major *mathematical* difficulties were encountered during the experiment. All groups succeeded in solving the problem – finding why the game was not fair and changing the rules to make it fair. The main mathematical teaching objective of the teaching sequence was fulfilled.

As to the *language used for communication*, obstacles in the teaching of those children whose mother tongue was not the language of instruction occurred. In the following text, the characteristics based on the receptive and productive nature of children's behaviour is presented. The results of the analysis and classification of interference from the perspective of didactics of mathematics in the CLIL environment are summarised. We also attempt to give hints for the teacher to avoid or minimize the influence of the detected interferences.

Learners' receptive skills

(a)   Unknown vocabulary

(a1) <u>New mathematical concept or symbol.</u> n such case, the learner does not understand the meaning, principle, rule.

Example: There were some special mathematical terms that were new to children, such as *probability, even chance, or sum*.

> *Experimenter: What is the more probable … you know the word probable?*
> *Pupil: – (Does not say anything.)*
> *E: More probable is more likely, you have bigger chance. For example: it is more probable that you will see a dog this afternoon, but it is not very probable that you will fly to the moon next week…you have a good chance to see a dog but almost no chance to go to the moon. So, what is more probable, more likely that you will get on the two dices? The sum, the total of ten or of six? … and why?*
> *P: (after a while looking at the diagram) The sum 6, there are more ways how to make it.*
> *E: Yes, there are more possibilities to make six.*

When preparing the lesson, the teacher should foresee what would be difficult for the learners. When introducing new vocabulary or special mathematical terms, s/he should always try to paraphrase the items, to illustrate them by examples, to use visualization, to demonstrate how they work and if necessary try to simplify. When preparing the lesson, it is highly recommended that the teacher prepares a list of key concepts and symbols with examples.

(a2) <u>Unknown vocabulary from ordinary language</u>. In this case, the learner does not understand the question, explanation, assignment, etc. although s/he is able to deal with its mathematical background successfully in other cases. The lack of everyday language may result in refusing to solve the problem or in misunderstanding of the task.

Example: In the last sentence of the previous example the answer of the pupil was repeated and the word *possibility* used instead of *ways*. It was a simplification, but did not lead to misunderstanding in this case. It was a brand new experience for the teacher that even with very limited knowledge of English, the communication with immigrant children was possible. The means of communication was automatically supplemented by language gestures, mimics, and pictures, repeating and showing on visual models.

When detecting this interference, the teacher is in a more difficult position. Usually, it differs from learner to learner and the teacher's help should be individualised. The teacher has several options how to react: e.g. modify formulations, simplify, use "everyday life" and visuals, and prepare written materials for learners with difficulties with language comprehension. Simplification can help the learners to understand, but the teacher must be aware of the misunderstanding that it can sometimes cause.

Language misunderstanding increases with the decreasing level of learners' language competencies. A necessary condition to succeed in cases of very limited language competencies is the presence of strong interest of the learner for the topic, practical use of the problem and activating teaching methods.

Remark: However, it is often very difficult to determine whether the learner did not understand the language itself or the mathematics in it.[3]

(b) Demands on concentration, thinking in a foreign language

This issue is discussed in (Marsh, Langé, 2000): "It is possible that the CLIL class may be perceived as 'more demanding' by the child, for the simple fact that listening, reading, speaking in an additional language is tiring until we get used to it. Therefore it is possible that the workload will feel heavier for the child, but it is up to the school to ensure that this is kept. …"

In CLIL, articulation, pace, intonation etc., are extremely important. Our experiments clearly manifest this point. Teachers have to pay attention to clarity of their speaking and adapt the pace to the level of the learners' language.

(c) Cultural interference

Another source of interferences mentioned in literature about second language acquisition and about bilingual education is the cultural dimension. We have not noticed this interference in our experiment. The reason could have been that the topic

was not related to a specific culture reality and also that we did not use authentic textbooks but materials that we prepared ourselves.

Nevertheless, to eliminate cultural interference it is recommended to collect information about the language environment of the learners and of the language of instruction to understand the culture differences better. When using authentic textbooks, one should realize and possibly draw learners' attention to cultural differences. (Novotná, Moraová, 2005)

(d) Verbalism and formalism of new piece of knowledge

By verbalism and formalism in this context we understand the situation when the learner knows the symbols and terms but does not understand their meaning. The new piece of knowledge is not included in the existing cognitive system, is not stable. (Hejný et al., 1990)

In our experiment, verbalism and formalism did not occur. We see the reasons for it in the following fact: The lesson was conducted in the form of a game (motivation). The teacher explained new vocabulary through exemplification, synonyms, which offered children the opportunity to create meanings of new notions and connect them with already known ones. The goal of the lesson was not to verbalize new knowledge but to create new game rules which naturally asked for relating the new ideas with the already known ones.

Learners' productive skills

In (Langé, 2002) it is stated that "limited language production of the pupils is a natural phenomenon especially by young learners". The goal of teaching mathematics through a foreign language is mainly to communicate certain knowledge, not to give enough language output. That's why we suppose the limited language production of learners is not something that the teacher should be anxious about too much.

(e) *The limited learners' language production* might have the following main consequences: Learners

(e1) have difficulties to formulate (correctly) an idea, pose questions and answer them, communicate clearly in the foreign language what was not understood;

(e2) make mistakes in grammar, syntax as well as use of vocabulary (in both ordinary and mathematical language);

(e3) are passive in oral communication.

Our experiments confirmed that the limited learners' language production in a session resulted in very low learners' participation in the talk and on the other hand, very high teacher's participation. The danger of presence of Topaze and/or Jourdain effects (Brousseau, 1997)[4] increases. The teacher either "helps the learners too much" or overestimates the quality of their knowledge.

(f) *The limited learners' knowledge of the language of mathematics*

It is manifested in a similar way as in (e). Code switching and discontinuous learner's speech are the common accompanying events. Often, learners substitute the correct term by its approximation from everyday life language or from the already known, but less precise, mathematical terms.

Example: The teacher asks children to describe the graph they obtained when recording graphically the results of throwing dices. Children do not know the terms increasing and decreasing function. They replace these terms by other ones with which they are familiar.

> E: *Actually, yes. How does the look …does how does the graph look like?*
> A: *eh...like a …*
> D: *A shade.*
> A: *No, like ....Like "I-sign"*
> E: *Like what, like I?*
> A: *????? ??????(murmurs)*
> E: *Different "Is"?*
> A: *Yes. Going down…At first it goes down … and then it is going … up.*
> D: *Down and up, up and down.*

The young learner speaks in short phrases and simple sentences often making many grammar, word order and usage mistakes. Mistake making is a necessary learning process and leads to language fluency. The sentences are shorter, and not as complex as adult-to-adult-speech. The teacher should rephrase and use repetition more frequently and check frequently that the learner has understood the message. Body language and visual reinforcement are emphasised when speaking to the young learner. The importance of the listening stage should not be overlooked and initially at the start of a CLIL programme the teacher produces most of the language. The teacher should know this and should not consider limited language production of his/her pupils as a sign of failure of bilingual learning.

6. CONCLUDING REMARKS

CLIL involves a number of issues. Marsh (1997) maintains that it is a motivating and challenging way of learning. By offering the target language as a tool, and giving the learner the opportunity to 'learn by doing', it is possible to reach positive and worthwhile outcomes. At the same time, it puts additional demands on the teacher related to the presence of three languages – mother tongue, foreign language and language of mathematics. In (Marsh, Langé, 1999), the following CLIL specifics are presented: the need of using a variety of media to bring the foreign language in the classroom, the role of redundancy or the ratio between teacher's and learners' talk volume, checking of comprehension or context-specific methodologies such as co-operative working styles are some of them.

The experiments showed that the teacher's limited language competence in CLIL interactions did not obstruct successful realization of the lesson. However, the preparation of the teaching episode was extremely demanding. The teacher's immediate self-reflection, analysis of setting up activities, elicitation of feelings, attitudes and emotions of pupils, focus on the use of texts and other materials and learners' production play the key role in the development of successful teaching strategies and language mastery of the teacher.

In CLIL classes, the teacher should be sensitive to the learners' needs as regards learning content, the mother tongue and the foreign language. S/he cannot prevent or at least diminish the interferences caused by the limited language skills of learners if s/he is not aware of the possibilities of their occurrence. In this we see the utmost importance of the analysis of the possible interference on the communication part of which is presented in this paper. We hope to contribute to the increasing sensitivity of teachers to these interferences, their nature and consequences.

REFERENCES

Brousseau, G.: 1997, *Theory of Didactical Situations in Mathematics*. Kluwer Academic Publishers, Dordrecht/Boston/London.

Brousseau, G.: 2002, *Cobayes et microbes*. Electronic discussion.

DeCorte, E.: 2000, 'Marrying Theory Building and the Improvement of School Practice: A Permanent Challenge for Instructional Psychology', *Learning and Instruction* 10, 249-266.

Forman, E.: 1996, 'Learning mathematics as participation in classroom practice: Implications of sociocultural theory for educational reform', in L. Steffe, P. Nesher, P. Cobb, G. Goldin and B. Greer (eds.), *Theories of mathematical learning*, Lawrence Erlbaum Associates, Inc., Mahwah, NJ, pp. 115-130.

Hejný, M. et all.: 1990, *Teória vyučovania matematiky* [Theory of mathematics teaching], 2. ed., SPN, Bratislava. (In Slovak.)

Hofmannová, M., Novotná, J. and Moschkovich, J.: 2004, 'Working with theories from outside mathematics education', in M.J. Høines and A.B. Fuglestad (eds.), *PME 28*. Eds. Bergen University College, Vol. 1, pp. 229-235.

Hofmannová, M. and Novotná, J.: 2005, 'Mathematics in a foreign language. Learning strategies', in A. Gagatsis, F. Spagnolo, G. Makrides and V. Farmaki (eds.), *Proceedings of the 4ᵗʰ Mediterranean conference on mathematics education MEDCONF 2005*. University of Palermo, pp. 137-146.

Langé, G. (ed.): 2002, *TIE-CLIL Professional Development Course*. Milan: M.I.U.R.

Marsh, D. and Langé, G. (eds.): 1999, *Implementing Content and Language Integrated Learning. A TIE-CLIL Foundation Course Reader*. University of Jyväskylä.

Marsh, D. and Langé, G. (eds.): 2000, *Using languages to learn and learning to use languages*. University of Jyväskylä.

Moschkovich, J. N.: 2002, 'A situated and sociocultural perspective on bilingual mathematics learners', *Mathematical Thinking and Learning*, Special Issue on Diversity, Equity, and Mathematical Learning. N. Nassir and P. Cobb (eds.), 4(2&3), pp. 189-212.

Novotná, J., Lebethe, A., Rosen, G. and Zack, V.: 2003, 'Navigating between Theory and Practice. Teachers who Navigate between their Research and their Practice. Plenary Panel', in N.A. Pateman, B.J. Dougherty and J. Zilliox (eds.), *PME 27/PME NA 25*. University of Hawai'i, College of Education, Vol. 1, pp. 69-99.

Novotná, J. and Hofmannová, M.: 2003, 'Context-dependent Learner Comprehension Strategies. Mathematics Taught in English to Czech Learners', in J. Novotná (ed.). *SEMT 03*. UK-PedF, Praha, pp. 117-120.

Novotná, J. and Moraová, H.: 2005, ,Cultural and linguistic problems of the use of authentic textbooks when teaching mathematics in a foreign language', *ZDM*, 37 (2), pp. 109-115. http://www.fiz-karlsruhe.de/fiz/publications/zdm/zdmcont.html

Pavesi, M., Bertocchi, D., Hofmannová, M. and Kazianka, M.: 2001, *Insegnare in una lingua straniera*. Milan: M.I.U.R.

Petrová, J.: 2005, 'Communicational obstacles and specifics in bilingual teaching/ learning of mathematics caused by the limited language proficiency of the children and of the teacher', in J. Novotná (ed.), *SEMT 05*, UK-PedF, Praha, pp. 268-276.

Walsh, S.: 2006, *Investigating classroom discourse.* London and new York: Routledge.

----

[1] In (Jirotková, Kratochvílová, 2004), the term communicational conflict is used. The authors speak about it when each participant of conversation understands the same word in different ways. The interference in our sense is broader, we admit e.g. the case when the participant has no idea etc.

[2] *The rules of the game:* You need two dice and some counters. Play this game with a friend. Take turns to roll both the dice and add the two scores. If the sum is 2, 3, 4, 10, 11 or 12, the first player takes a counter. If the sum is 5, 6, 7, 8 or 9, the second player takes a counter. The winner is the first to gain fifteen counters.

[3] This problem is discussed in the work of Hofmannová, M., Novotná, J., Pípalová, R. (2004). Assessment Instruments for Classes Integrating Mathematics and Foreign Language Teaching. *ICME 10, TSG 27*, Copenhagen.

[4] Topaze effect: The teacher begs for a sign that the student is following him, and steadily lowers the conditions under which the student will wind up producing the desired response.

Jourdain effect: It is a form of Topaze effect. The teacher … claims to recognize indications of scholarly knowledge in the behaviour or responses of a student, even though they are in fact motivated by trivial causes. It is a form of the Topaze effect.

A QUESTION OF AUDIENCE, A MATTER OF ADDRESS

David Pimm, Ruth Beatty and Joan Moss

(University of Alberta, University of Toronto, University of Toronto)

This paper draws attention to certain linguistic features present in notes posted to a common computer data base (Knowledge Forum) by grade four, five and six students in two schools working on a set of mathematical generalising tasks. The notes exhibit some elements present in various sorts of student mathematical writing familiar from other contexts (for instance, accounts of classroom-based mathematical problem solving). One particular feature that this setting might plausibly seem to accentuate is that of the addressivity of the writing, namely the 'turning toward' the other, a feature that is singularly absent from more formal mathematical prose. Nevertheless, despite the relatively young age of these students, their writing also exhibits instances of more 'sophisticated', apparently unaddressed writing.

One of the more taxing questions implicated in the complex interrelationship between language and mathematics has to do with the shaping of form by content and of content by form. One of the less considered aspects of this mutual influence has to do with the nature and influence of the *audience* for the language, especially written mathematical language where the empirical reader (one possible but by no means exclusive audience) may not be co-present with the author, either temporally or spatially. Yet, as Bakhtin (1952/1986) was insistent in claiming, every human utterance is addressed *to* someone, a phenomenon he termed *addressivity*, namely an orientation toward the other.

> An essential (constitutive) marker of the utterance is its quality of being directed to someone, its *addressivity*. As distinct from the signifying units of a language – words and sentences – that are impersonal, belonging to nobody and addressed to nobody, the utterance has both an author (and, consequently, expression as we have already discussed) and an addressee. This addressee can be an immediate participant-interlocutor in an everyday dialogue, a differentiated collective of specialists in some particular area of cultural communication, a more or less differentiated public, ethnic group, contemporaries, like-minded people, opponents and enemies, a subordinate, a superior, someone who is lower, higher, familiar, foreign, and so forth. And it can be an indefinite, unconcretized *other* […] (p. 95)

One such question of audience signalled by our paper's title, then, concerns the addressivity of a mathematical text, which is not always a straightforward matter, as John Fauvel (1988) has observed:

> Euclid's attitude [towards the reader] is perfectly straightforward: there is no sign that he notices the existence of readers at all. […] The reader is never addressed. (p. 25)

There are overly-common presumptions about whom a student writer is writing for: for the teacher, for the examiner, for her- or himself, for posterity, for Bakhtin's indefinite other, … . Umberto Eco (1979) has written insightfully about the *model*

author and *model* reader in relation to pedagogic texts (or at least ones with an arguably pedagogic function among others), contrasting it with the empirical reader, say you or me. (For more on this notion in a mathematics education context, see Love and Pimm, 1996.) While these notions are certainly useful in analyzing adult-authored texts intended for students of various ages, it is less clear that they apply to neophyte texts produced by young children, whose awareness of some of the sophistications of and conventions concerning authorship may at best be described as emergent.

1. SOME CONTEXTUAL DETAIL

In this paper, we would like to explore some aspects of mathematical audience by means of written data obtained from grade 4, 5 and 6 students in two Canadian schools, whose classrooms were connected by means of a software environment called *Knowledge Forum* (Scardamalia, 2004; Scardamalia et al., 1994; Moss and Beatty, 2006; in press). [1] This paper reports on part of a considerably larger investigation, among other things comparing spoken, face-to-face discussion of mathematical problems with these written counterparts. Due to space, or rather its absence, we focus here only on the written texts.

Knowledge Forum is a networked, multimedia community knowledge space created by community members, in this case *unsupervised and unmediated by the teachers or others*. By authoring *notes*, participants contribute ideas (theories, conjectures, patterns, working models, claims, evidence, data, and so forth) to *views*, which are workspaces for clusters of related activity carried out by the classroom community. Students can either contribute their own notes or co-author them, and have the means to respond to or build onto one another's ideas. Knowledge Forum also has customisable '*scaffolds*'. Examples include: "My theory", "I need to understand", "New information", "This theory cannot explain", "A better theory" and "Putting our knowledge together". These theory-building scaffolds are intended to encourage participants to enter, improve and search community accounts of ideas (Scardamalia, 2004). Activity in the database (reading, writing, building on, etc) is recorded automatically.

Figure 1 on the next page presents one of the views from this study, whose notes were generated in response to the Perimeter Problem by students from two connected grade-four classrooms. The small squares represent student notes, the connecting lines represent *build-ons* created as students read and respond to each other's contributions, thus providing a network expressing connectivity, one which also codes relative chronology (a note written before or after another) within a conversational thread.

The data-base views are continuously evolving interactive discourse spaces, where each thread of conversation on a problem is documented, webs of interchanges graphically displayed (see the top part of Figure 1 below: this view is what the students actually see for each problem and each note is labelled with the author's

name and the first few words of the note itself), and collective understandings captured. Two different general data bases were used: one for grade 4 and one for 5/6 students.

Figure 1: Problem View and a Note

Each note contains a space for composing text (or graphics) and a list of meta-cognitive scaffolds. The sample note in the lower part of Figure 1 was authored by a student who contributed a solution to the Perimeter Problem. In this particular note, she used the introductory genre scaffold *My theory*.

Students can also use Knowledge Forum's graphics palette to create illustrations or they can scan in drawings, function tables or photographs to support their explanations. These visual representations have two purposes. First, they serve as tools for problem solving; second, they provide students with the means to illustrate and elaborate on their theories. The data base provides a permanent record of each student's thinking; because the discussion is asynchronous, students can revisit their own conjectures or the theories of others at any time.

The 142 participating grade 4, 5 and 6 students came from two schools, an inner-city public school (grade 4, grade 6 classes) ranked as the third most 'needy' by the school district due to a high ESL, low SES population (School A) and a laboratory school (with a grade 4 and a 5/6 class) – School B. The students from the two schools did not know each other and came from different backgrounds both from the perspective of demographics and mathematics instruction, the former school being more traditional and the latter more reform oriented. Prior to working on Knowledge Forum, all students engaged in an extended series of classroom experiences with geometric growth patterns and their expression as part of the intervention study. The generalising problems posted on Knowledge Forum presented students with patterns of growth in different contexts. For each one, students were asked to find the underlying structure and express it as an explicit function or "rule".

The problems that the students worked on were different for the two grades and included linear and quadratic rules embedded in different contexts. All were chosen for this study as a means of developing the student's reasoning about functional relationships.

2. LOOKING AT NOTES THROUGH THEIR DISCOURSE FEATURES

Various discourse features have been identified in terms of their salience for mathematical text, in particular pronouns, deictic markers, hedges, genre elements, verb tense, forms of politeness and modal elements (for a broad survey of this area, see Pimm, 2006; see also Morgan, 1998). One of the complexities of this sort of work is that many of these features are not independent of each other; they co-occur and frequently interact. As a small instance, it has become a commonplace observation that a student expression changing from 'I did' to 'you do' can signal a shift from a narrated account of particular, personal temporally and spatially located experience to an attempt to express a more generalised observation (a linguistic reflection of the triple processes of decontextualisation, depersonalisation and detemporalisation to which Balacheff (1988) draws attention). This specific instance involves the interaction of a tense change with a pronoun switch. Similarly, variation in modality can be compounded with issues of politeness and hedging.

In this study we draw on the above strand of discourse-analytic work for our analytic tools, as well as selection of general linguistic phenomena. As such an approach to the data focusing on the phenomenon of addressivity is somewhat exploratory, we remain both close to and open to the specific data we are exploring.

In terms of the data generated by this study, in addition to the use of proper names, we attend specifically to pronominal features (including their absence) as an indication of both intended author and addressee deixis, as well as other means by which a reader may discern the locus of address. A number of pronouns are 'general' or underdetermined (see Netz, 1998, for a discussion on the notion of 'underdeterminedness' in a mathematical context) in terms of their referent. We are also interested in the 'tone' of the address (see Fauvel, 1991, or Bills, 1999).

We start with some general observations about the two classroom pairings. In general, the individual notes from the grade 5/6 classes were more extensive. However, initially at least, the connectivity in the views remained intra-classroom. The grade fours however, went for shorter cross-classroom notes right away. There was also a noticeable difference between initial postings (which we term *originals*) and any subsequent responses (which we call *build-ons*). Arguably, the author of an initial posting is more aware of writing to the whole community as audience for the note, some of whom they know and others whom they know they do not. "Everyone is going to read this" may be uppermost in the mind or the writer of an original. The subsequent build-on notes in a loop often have greater specificity of address (whether explicitly marked, e.g. "I disagree with you Jessica" or not), even though they are responding to an individual posted message publicly. (This is in contrast with Phillips and Crespo's 1996 work involving grade fours writing a sequence of 'pen-pal' letters to university pre-service elementary teachers, where apart from the teacher-researcher there was no known public audience other than the addressee.)

Many original notes have a certain formality and show an intention of attempting to be very clear and straightforward in presenting an account (and are usually accompanied by some kind of rationalisation, explicitly signalled by the word 'because'). They are centripetal (see, for instance, Dowling, 1998), that is focused in towards the author and what is to be expressed and need to be centred and self-contained. They contain fewer indeterminate pronouns or other extra-textual elements. By contrast, the response notes are generally more informal and tend to be more specific in focus and in various ways reach out or back to previous authors and their notes.

There were also a number of explicitly hedged elements (see, for instance, Rowland, 2000) in the notes, where the author marked his or her uncertainty about or tentativeness of commitment to a claim or assertion being written (the problem involves trapezoidal tables and the number of people who can sit at a growing line of them). As might be expected, they occurred most in originals. The example below (written in response to a problem that asks for the number of people who can sit at

any number of trapezoidal tables placed end to end) starts "I'm not sure" (which is not a system support) and repeats three times 'I think' something is the case. Nevertheless, the generalisers 'every time' and 'each time' are used, suggesting a mathematical awareness of the problem's intent.

> **Add on** – ES (B)
> I'm not sure but i think that every time there is another table it adds on 3 because if there is 1table and it=5 and then when it is 2 tables and it=8 and for 3 tables it=11 and for 4 tables it=14
> and for 5 tables it=17.
> So...I think each time they add on 3!
> So I think the rule is:+3

Each note has the scope for a title and an author, as illustrated in our first example above as well as in the note in Figure 1, which is labelled 'Drawing and t-chart' and authored by Rachelle (a grade 4 student from School A). If we look at the language of the sample note in Figure 1, there are a number of discourse features evident. In keeping with the single name in the author line, the prompt 'My theory' seemingly individualises the note's voice to that of a single author ('Our theory' is an alternative support-prompt choice), but very quickly we find reference to a second textual presence 'my partner Janine', though we do not know whether or not she was physically present when this particular note was being written. Nevertheless, attribution of work reported in evidence for the offered rule was to the pair 'how we figured out the rule'.

In keeping with conventional mathematical style (see, for instance, Solomon and O'Neill, 1998), the mathematical claim about the particular rule is made in the present tense (as is the claim of this being 'my theory'), while everything that follows the 'because' is in a narratively-structured simple past-tense account, what Marks and Mousley (1990), following Martin, refer to as the 'report' genre. [2] The chronology within the account is a little confused (e.g. the sequences marked by 'when', 'after' and 'then' does not seem to indicate a uniform time line). In terms of its content, it is unclear from what was presented whether this is an inductive generalisation based upon the systematic exploration of five consecutive cases (what Balacheff, 1988, would characterise as 'naïve empiricism) or whether one of these instances could be talked through as a 'generic example'. This note was posted with the intention of presenting a rule and a rationale, and was neither written in response to another nor followed up upon by others.

There is no apparent explicit addressivity: there is no salutation at the beginning (as, for example, there would be with a 'friendly letter' a common genre taught in grade 4), nor apart from the mention of her partner Janine are there any deictic pointers to others (e.g. specific readers, the Knowledge Forum community, …).

> **The Rule** – Uri (School B)

the rule is x3-3. you times   the sides and - the corners beacause the corners are not shared within the sides.

Who is right? – Missy (School B)

But whats his name said that it was times 4 plus 1! So whitch one is right? Yours or whats his name?

Other rules – SF (School A)

i agree but you know their could be other rule's like their was in the last view

yes – AK (B)

I agree cus I found 2 rules that work so far.

Different rule – Uri (School B)

there is another rule but explains x3-3 in a diferent way.

i think its actualy number + (n-1) (number-1)+ (n-2)

make sense – Finn (School B)

I agree with you

In this exchange, the expression 'Whats his name' suggests a certain address (as the referent indicated is part of the community), but is used between two members of the same class about someone who is not. Missy seems to be assuming everyone has read all the contributions, so the referent will be clear. That said, the referent for the final 'you' being agreed with is unclear. This is because although there is a temporal sequence of utterances, there are none of the paralinguistic features of a face-to-face conversation (such as gaze direction or gesture) to assist disambiguation.

In the next linked pair of notes (between two grade 6 students from the same classroom on a different problem), direct address is in evidence, as is a certain pronoun turbulence.

How to figure out this problem – Krishnendu (School A)

My theory is you could multiple column by row togive you an answer but you can dived 2 since the staircase is in half.

Disagreement – Chograb (School A)

I disagree with your theory Krishnendu. IT make a lot of sense but for the 4th position the number of blocks in rows = 4 and number of block in columns is = 4. So 4x4=16 and when you divide that by 2 you get eight. But we need to get 10. I think you are on the right track.

The first note involves a non-standard switch between the modals 'could' and can', but the use of 'you' is in the general personalised equivalent of 'one'. In the second note, there is an interesting double deictic use of the pronoun 'you': Krishnendu is directly addressed and the specific reference of 'your' in 'your theory' is subsequently made clear. The first 'you' in when 'you divide that [16] by 2 you get eight' we claim is the general 'anyone' use common in mathematics. Then the switch to 'we' in 'But we need to get 10', where there is a common project being (this both is and isn't Krishnendu's problem) and then back to very personal specifics of direct

address, where Chograb ('I') makes a teacher-like comment to Krishnendu ('you') in 'I think you are on the right track'.

Phillips (2001), in her extensive study of grade-four mathematical writing in a variety of non-standard settings, found her students frequently making such encouraging 'teacherly asides' as part of the emotional tenor of the directly-addressed feedback on some tasks (e.g. following a set of child-created instructions to make a specific, pre-determined shape from pattern blocks), as if this were a required or desirable feature of classroom writing. It could also be seen as a form of politeness (as, arguably, can 'IT makes a lot of sense'), an attempt to mitigate somewhat the force of the on-the-record, face-threatening act of explicit disagreement announced both in the note title and the opening words. (For more on Levinson's account of politeness invoked in a mathematics education context, see Bills, 2000.) Although 'I disagree' is a Knowledge Forum scaffold, students rarely used it explicitly as such, despite frequent overt disagreements.

Our last example illustrates another welter of short responses to an initial posting (the problem concerns the number of faces showing when a rod of *n* cubes is created).

> **Smiley stickers problem** – Mike (A)
> My theory is that you would need 5 stickers for 1 cube so for 2 cubes you would have 10 stickers. But for 3 cubes you would have 15 stickers.
> **How many stickers?** – Frank (B)
> Can you explain your note better I don't really get what you are saying.
> **Smile stickers** – Jenn (A)
> I diagree because a rode of one cube is equal to 6 stickers and the second one was right.
> **Stickers** – Givegga (A)
> I disagree with you because a cube has 6 faces so you should have 6 stickers.
> **6 sides for 1** – Ella (B)
> I don't agree because if there was one cube then there would be 6 stickers because one cube has 6 sides because on on the bottum one on the top and four on the sides
> (if you don't get what I'm saying look at this okay, also I'm drawing the cubes sides making the cube ok):
> **do not count** – Shauna (B)
> there are 6 stickers on 1 cube but when you put another cube it makes 10 because the stickers that are beside eachother do not count
> **six sided cube** – SRW (A)
> My theory is that on one cube there would be 6 sickers eecause there are 6 sides.
> **1 cube six sides** – JI (A)
> With 1 cube there would be 6 sides. With 2 cubes there would be 10 sides. With 3 cubes here would be 14 sides. And on and on.

A full analysis of these exchanges is beyond the scope of this paper, but the pronouns alone make for an interesting challenge in keeping track of reference, to say nothing of the claims, counter-claims and reasons offered in response to the first post. The subtle use of modals ('should', 'would') intermingled with declarative utterances ('it

makes 10', 'there are 6 sides') and the confident uses of 'because' make these exchanges rich sites to examine students' emergent argumentation forms.

IN SUMMARY

By means of exploring a range of student responses generated in a particular, computer-mediated communicative setting (Knowledge Forum), we have been able to highlight instances of addressivity between the texts, where an author leans towards her audience in more or less explicit and tacit ways. However, Bakhtin's claim about addressivity being a core element of any communication raises the question of why in more formal mathematical prose such elements are systematically absent. We are left wondering to whom such a denatured mathematical utterance-text is addressed, stripped of pronouns and time and other specific ties to the deictic world of the human here-and-now. Students working in this environment reveal some inclination toward explicit addressivity of their notes about their mathematical claims, theories and findings. But they also, already in grade 4, show a certain attunedness to more formal mathematical conventions by which mathematical utterances are both marked and expressed. And finally, are we all fated, as mathematical readers, to remain, in the words of the poet Robert Kroetsch (1989), "eavesdroppers on an address to the sacred" (p. 163)?

[1] The image of the Roman *forum* is interesting, conjuring a place where people may come together freely to discuss matters of public or common import. It is also the origin of the term *forensic*, the sort of examination we intend to subject the student data to, in order to present in the public setting of CERME.

[2] This also fits well Bruner's (1986) dichotomy between *narrative* and *paradigmatic* styles.

REFERENCES

Bakhtin, M.: 1953/1986, 'The problem of speech genres', in C. Emerson, and M. Holquist, (eds), *Speech Genres and Other Late Essays*, University of Texas Press, Austin, TX, pp. 60-102.

Balacheff, N.: 1988, 'Aspects of proof in pupils' practice of school mathematics', in D. Pimm, (ed.), *Mathematics, Teachers and Children*, Hodder and Stoughton, London, pp. 216-235.

Bereiter, C. and Scardamalia, M.: 2003, 'Learning to work creatively with knowledge', in E. de Corte, *et al.* (eds), *Powerful Learning Environments: Unravelling Basic Components and Dimensions*, Elsevier Science, Oxford, Oxon, pp. 55-68.

Bills, E.: 1999, 'Students talking: an analysis of how students convey attitude in maths talk', *Educational Review* 51(2), 161-171.

Bills, L.: 2000, 'Politeness in a teacher-student context: a socio-linguistic analysis', *For the Learning of Mathematics* 20(2), 40-47.

Bruner, J.: 1986, *Actual Minds, Possible Worlds*, Harvard University Press, Cambridge MA.

Dowling, P.: 1998, *The Sociology of Mathematics Education: Mathematical Myths/ Pedagogic Texts*, Falmer Press, London

Eco, U.: 1979, *The Role of the Reader*, Indiana University Press, Bloomington, IN.

Fauvel, J.: 1988, 'Cartesian and Euclidean rhetoric', *For the Learning of Mathematics* 8(1), 25-29.

Fauvel, J.: 1991, 'Tone and the teacher', in D. Pimm and E. Love (eds) *Teaching and Learning School Mathematics*, Hodder and Stoughton, London, pp. 111-121.

Kroetsch, R.: 1989, '*Hear Us O Lord* and the Orpheus occasion', in R. Kroetsch, *The Lovely Treachery of Words: Essays Selected and New*, Oxford University Press, Toronto, ON, pp. 163-178.

Love, E. and Pimm, D.: 1996, '"This is so": a text on texts', in A. Bishop, *et al*. (eds), *International Handbook of Mathematics Education*, Kluwer Academic Publishers, Dordrecht, The Netherlands, pp. 371-409.

Marks, G. and Mousley, J.: 1990, 'Mathematics education and genre: dare we make the process writing mistake again?', *Language and Education* 4(2), 117-135.

Morgan, C.: 1998, *Writing Mathematically: the Discourse of Investigation*, Falmer, London.

Moss, J. and Beatty, R.: 2006, 'Computer-supported collaborative learning of patterns and generalizing problems in fourth grade', *International Journal of Computer Supported Collaborative Learning* 1(4), 441-465.

Moss, J. and Beatty, R.: in press, 'Knowledge building and mathematics learning: shifting the responsibility for learning and engagement', *Canadian Journal of Learning Technology.*

Netz, R.: 1998, 'Greek mathematical diagrams: their use and their meaning', *For the Learning of Mathematics* 18(3), 33-39.

Phillips, E.: 2002, *Classroom Explorations of Mathematical Writing with Nine- and Ten-Year-Olds*, Unpublished doctoral dissertation, Open University, Milton Keynes, Bucks.

Phillips, E. and Crespo, S.: 1996, 'Developing written communication in mathematics through math penpal letters', *For the Learning of Mathematics* 16(1), 15-22.

Pimm, D.: 2006, 'Discourse analysis and mathematics education', Lecture presented at ICME-10, *Proceedings of ICME-10*, CD-ROM.

Rowland, T.: 2000, *The Pragmatics of Mathematics Education*, Falmer, London.

Scardamalia, M.: 2004, 'CSILE/Knowledge Forum', in A. Kovalchick, and K. Dawson, (eds), *Education and Technology: an Encyclopedia*, ABC-CLIO, Santa Barbara, CA, pp. 183-192.

Scardamalia, M., Bereiter, C. and Lamon, M.: 1994, 'The CSILE project: trying to bring the classroom into World 3', in K. McGilly, (ed.), *Classroom Lessons: Integrating Cognitive Theory and Classroom Practice*, MIT Press, Cambridge, MA, pp. 201-228.

Solomon, Y. and O'Neill, J.: 1998, 'Mathematics and narrative', *Language and Education* 12(3), 210-221.

SELECTED PROBLEMS IN COMMUNICATION BETWEEN THE TEACHER AND THE PUPIL EXPLORED FROM THE SEMIOTIC VIEWPOINT

Filip Roubíček

Institute of Mathematics of the Academy of Sciences of the Czech Republic, Prague

Signs help us to grasp and understand abstract mathematical objects and relationships among them. For communication in mathematics, it is important to acquaint the pupils not only with various semiotic systems, but also with the rules how to form, interpret, and use them properly. Researches show that if the mathematical language is not developed on all of its levels and if analyses and writing of problem solutions are performed formally only, numerous problems appear in the communication.

Hejný and Kuřina (2001) have called attention to school conformism – assumption of teacher's formulations when the pupils do not develop any representation of knowledge. Duval (2001) added that understanding in mathematics implies coordination of at least two semiotic representations.

The difference of images (mental representations) and contexts tends to be one of the most frequent causes of misunderstanding in the communication between the teacher and the pupil. It is especially up to the teacher to take this fact into account during communication, and to eliminate the misunderstanding by choosing suitable language means. A common language in which the concepts used have a very close content for the teacher and the pupils, and in which the words connotate similar meanings in their minds, is very important for the communication.

A significant role in words and symbols interpretation is played by the communication context, i.e. the framework within which the communication takes place. The context is determined by images of the pupil and the teacher that affect their understanding and usage of communication means, by the social environment, and cultural customs. The phenomenon when the pupil uses an expression that does not correspond with the communication context or when the pupil uses such an expression in two different semantic contexts is called the communication confusion. Communication dissonance is a phenomenon caused by communication confusion that leads to discordance or disagreement between the communicants.

REFERENCES

Duval, R. (2001). The Cognitive Analysis of Problems of Comprehension in the Learning of Mathematics. In *Proceedings of PME 25*. Utrecht.

Hejný, M. and Kuřina, F. (2001). *Dítě, škola a matematika: konstruktivistické přístupy k vyučování (Child, School and Mathematics: Constructivist Approaches to Teaching)*. Prague: Portál.

OBSTACLES IN MATHEMATICAL DISCOURSE DURING RESEARCHER-STUDENT INTERACTION

Jana Slezáková, Ewa Swoboda

Charles University in Prague, Faculty of Education, Czech Republic
University of Rzeszow, Institute of Mathematics, Poland

One of the characteristic phenomena of class communication is misunderstanding. We will describe moments of mutual misunderstandings in communication during our experiments. By analyzing and labeling them, we will illuminate some obstacles in mathematical discourse.

RATIONALE

Each pupil has his/her own experience. Communication mediates the exchange of this experience and thoughts in general. The consequence of this exchange is the pupil's richer and more precise understanding of mathematical ideas and metacognitive knowledge of different approaches to one concept, procedure, problem, etc. (Bussi, 1998, Steinbring, 2005, Kieran et al., 1998). The quality of class communication and interaction depends not only on the teacher's ability to create kindly working climate but first of all – on ability of effectively lead discussions. However, an unavoidable attendant phenomenon of class communication is misunderstanding. Vygotsky says: *The speed of oral speech is unfavourable to a complicated process of formulation – it does not leave time for deliberation and choice. Dialogue implies immediate unpremeditated utterance. It consists of replies, repartee; it is a chain of reactions.* The main aim of our research is to describe moments of mutual misunderstandings in mathematical communication between an adult and a pupil. By the research, it became clear that mutual understanding depends on many aspects that overlap.

This phenomenon cannot be accepted as undesirable because misunderstanding motivates communication and looking at one's own knowledge from the outside, revising it, making it more precise and enriching it. It is also important to realise that teachers should allow for potential misunderstandings, and their reactions should be guided by the "do we understand each other?" attitude, rather than by the conviction that "the student is wrong".

THEORETICAL FRAMEWORK AND RELATED LITERATURE

Theoretical base for our research is Vygotskian approach, elaborated mainly in the book "Thought and Language" (1989). Additionally, we link our work with publications concerns the psychology of communication.

Misunderstandings in communication have various roots. A theory of communication, as the art of reaching mutual agreement, lists series of barriers to

human understanding (Mckay, Davis, Flanning, 2004). The measure of understanding obviously depends on coherence of partner's experiences. In another word it depends on the "area of overlapping" of their personal experiences: more overlapping – better understanding. In general, if people have different experience, the same word can have different meanings for them. This observation has a fundamental consequence for the process of teaching mathematics, where words and symbols have a special importance.

A teacher has his/her own individual cognitive structure (Hejný, 2004), which is, among other things, reflected in the language he/she uses. It is obvious that a teacher's structure is very different from the structure created in student minds. The differences between students' and teachers' cognitive structures generate different relationships and interconnections between facts, which can be the main reason for misunderstandings between them. The core of misunderstanding lies in the divergence of interpretations of words, pictures, graphs, etc. in the consciousness of individual minds. According to the fact that the core of misunderstanding is a difference in experiences fields of teacher and student, it is impossible avoid misunderstanding by any easy way.

METHODOLOGY

The topic of our research is misunderstanding which emerges spontaneously in mathematical discourse. Thus it cannot be investigated in planned experiments. A researcher interested in this problem has to look for this phenomenon in his or her own previous experiments (done for other purposes), or analyse the observed misunderstanding in class interaction between the teacher and the student. We have chosen the first option. It gave us the chance of reliably analysing the attitudes of at least one of the individuals taking part in the observed situation, by practising self-critical reflection. When conducting some experiments in the past, we felt that there were some moments with disharmony in the researcher-pupil interaction. We chose such protocols and analysed them. In this paper, we present some results of our work and illustrate them by one example.

Our research has consisted of the following stages:

1. Choosing protocols and discussing them. We analysed two research sessions conducted by each of the authors. Our first approach to get in the research problem was to describe each experiment from the author's individual point of view. On the basis of our discussions a need for common language appears.

2. Building common language. Building common language was based on analysing the research sessions and identifying phenomena. At the beginning of analysis we distinguished the positions of two communicants taking a part in the experiment. It was a student and researcher. The core of misunderstanding we would like to study lies in a different interpretation of the same words, idioms, sentences, ... in students' and researchers' minds. To describe this misunderstanding it was necessary to

analyse deeply relevant fragments of the protocols. In the analysis, we used the method called Atomic Analysis (Hejný, 1992). The method is based on the decomposition of a student's written work or an interview protocol into the smallest meaningful parts called static atoms. These atoms are then used to create hypothetical dynamic atoms which are mental steps connecting one static atom to another. Dynamic atoms identified within our analysis were classified into cognitive, social and emotional spheres. This classification was developed from a well known classification *social* (Bussi, 1998; Steinbring, 2005) versus *individual* (Kieran et al., 2001). For our purposes, the individual spheres were divided into two parts *cognitive* (Hejný, 2004) and *emotional* (Evans, 2004; Nelmes, 2004). After distinguishing these spheres, we constructed a row-column table (Kratochvílová, Swoboda, 2004). This research tool was used for a common deep analysis of a research session.

3. Identifying misunderstanding moments and their analysis. The fulfilled tables give us the whole picture of interactions during the research sessions. From this we focussed only on misunderstanding moments, i.e. places where at the same time two (or more) dynamic atoms interacted and this interaction was a cause of misunderstanding. The relevant dynamic atoms gave us evidence how these misunderstanding moments appeared. The table was fragmentised. On this base we created new twelve tables in which only relevant dynamic atoms (nodes) were presented. Such tables were a base for re-analysis of misunderstandings. This re-analysis was aimed at deeper description of misunderstandings. In this way we created a new organization of the research material. Even though we analysed misunderstandings for each research session separately, it was a new start point for the next stage of our research.

4. Classifying and characterising obstacles. We took each of twelve tables separately and by labelling them we characterised misunderstanding moments as obstacles in discourse. Then we started to compare the obstacles in direction to find some common general labels covering at least two different obstacles from our two different research sessions. It was an impulse to formulate the labels as general as possible. It gave us the classification of obstacles. Having these labels we returned to our previous analysis and we re-analysed the obstacles from this new point of view.

5. Generalizing obstacles. At this stage we try to describe obstacles in general way without any particular connections to the concrete examples. We show that such obstacles exist in common discourse but for mathematics they have a special meaning. We try to formulate statements described obstacles in such way that these statements could challenge educators to look for concrete examples from school reality.

6. Relating obstacles to descriptions in literature. We tried to link the phenomena identified in our observations to some theoretical descriptions in literature. In particular, we were interested in the communication between an adult and a child. As

it turned out, the most significant theoretical source was the reference to Vygotsky's work.

DESCRIPTION OF TWO RESEARCH SESSIONS

1. Research session I

The aim of the research session was to investigate the ability of the student to use his/her mathematical knowledge of similar figures in atypical situations. The prepared situation allowed for understanding similarity in an everyday sense or in the sense of the mathematical definition (or mental model) of similarity. The research tool was a set of bracelets of different thickness and diameter and made from different materials.

A fourteen-year-old student Kuba had learned the definition of similar polygons and solids at school. Half a year later he took part in the session described below. Kuba knew researcher as a mathematics teacher, working at the same school, which Kuba visited. The session took place after school lessons, in one of the staff rooms.

Protocol:
1. Res.01: *(showing the set of bracelets)* Find out if these objects have any relation with the mathematical concept of similar figures.
2. Kuba 01: *(no doubt)* These are similar.
3. Res.02: Mathematically?
4. Kuba 02: Yes. From the mathematical point of view.
5. Res.03: *(with surprise in her voice)* From the mathematical point of view?
6. Kuba 03: Yes.
7. Res.04: ……………….?
8. Kuba 04: Because in everyday language they are not similar at all.

2. Research session II

The research aim was to analyse solving strategies for the so-called Abracadabra problem (Polya, 1966) used by pupils aged 8-9. This research session was conducted with a 9-year-old boy, Marek. The second author was as a researcher.

The student was given a sheet of paper, on which there were six identical maps in one row (one of these maps is beside). There were pens, pencils, rulers, clean sheets of paper and the sheets of paper with maps.

The problem was given to him orally as follows (The researcher had the text of given problem written on paper): There is a city plan on the figure. Find all the paths from the left bottom corner to the right upper corner (*Res. shows both corners.)* You can go only up or right (*Res. shows both directions.)*. If you take the same path twice you will pay fine. Find all the paths without paying the fine.
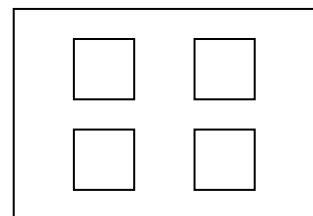
Before the research session the researcher decided not to interfere in the session. That is she will not disturb the student's work and comment it. Because of this reason there was no reach communication. The session took place in a staff room during one lesson.

Protocol (The researcher's notation of paths was: *r* means "right", *u* means "up" and the word *ruur* is the name of the path "right, twice up, right"):

1. *Researcher gives Marek the sheet of paper with the six maps.*
2. *Res. 01:* There is a city map in the figure. Find all paths from the left bottom corner to the right upper corner (*Res. shows both corners.*) You can go only up or right (*Res. shows both directions.*) If you take the same path twice you will pay fine. Find all the paths without paying the fine.
3. *Marek draws the first four paths: (1) rruu, (2) urru, (3) urur, (4) uurr.*
4. *Pause 2 minutes.*
5. *Marek draws the path: (5) ruur.*
6. *Pause 4 minutes.*
7. *Res. 02:* Look. (*Res. pointed with her finger at all the paths that have been drawn.*)
8. *Pause 1 minute.*
9. *Marek draws the paths: (6) ruru.*
10. *Res. gives Marek another sheet with maps.*
11. *Res. 03:* Look for the other paths in order not to pay any fine, please.
12. *M. 01:* ..... How much fine can I pay?
13. *Res. 04:* It depends how many times you will take the same path.
14. *Pause 1 minute.*
15. *M. 02:* I think that I have found all the solutions.

RESEARCH RESULTS

As a result of theoretical part of research we identified several phenomena, which exist as obstacles in communication. In this paper we concentrate only on these phenomena that have a cognitive characteristic:

1. Different understanding concerning context of situation
2. Focus on own aims
3. Different importance assigned to words in statement
4. Different understanding of the key word

We are of the opinion that these categories are not disconnected. Many real situations can be simultaneously interpreted from the multiple positions of two (or more) obstacles. The passage below lists a short presentation, in which we explain how these phenomena emerged in our research.

IDENTIFYING MISUNDERSTANDING MOMENTS. ANALYSING AND LABELING THEM.

1. Different understanding concerning context of situation. In the research session I, physical objects from real world were connected with a world „similarity", but this

connection was different for both participants. (Res. 01*: Showing the set of bracelets: Find out if these objects have any relation with the mathematical concept of similar figures).* The researcher took bracelets as three-dimensional solids and asked to recognize these models in the domain 'similarity'. The researcher knows that the mathematical concept 'similar' origins from everyday understanding of similarity as a result of schematisation. The things used in experiment had a lot of common characteristics: they were used for the same goal (hand decoration), they all had a "similar" round shape. But in mathematical terms – treated as three-dimensional objects – they were not similar: inner proportions of every object were different. For the student, however, the objects (bracelets) joined with the word 'similar' – in combination with the researcher being seen as a teacher by him – might have only been connected with the mathematical concept of similarity that he had learned at school where the emphasis was put on the similarity of two-dimensional figures. In the research session II (*Res. gives Marek another sheet with maps*) the research knows that if the task is to find *all* solutions there are two steps of solutions: 1. to find solutions, 2. to prove that no more solutions exist. Because the second step missed the researcher gave another sheet of paper as a challenge. However, the boy interpreted the researcher's gesture as: 'The teacher expects me to keep on working and find other paths.'

2. Focus on own aims. In the research session I (Res. 03: *With surprise in her voice: From the mathematical point of view?)* the researcher was so occupied by the aim – to find out if the boy understands similarity in everyday or mathematical sense - that she had not been able to interpret the information that he sent. In the research session II (Fragment 3: *Marek draws the paths (6) ruru. Res. gives M. another sheet with maps)* the researcher only needed to convince herself that the student knew that there were not any more solutions. The boy knows that he solved the task because he had fulfilled all maps. He did not expect any continuation concerning this task.

3. Different importance assigned to words in statement. In the researcher's task (the research session I, *Find out if these objects have any relation with the mathematical concept of similar figures*) the most important word is 'similarity' but for the boy it is 'mathematical'. It caused that the thinking processes of both, the boy and the researcher, were driven differently. The boy *made the projection* of real objects into mathematical word. The researcher *differentiated* mathematical meaning of word 'similar' and everyday meaning of word 'similar'. In the researcher's information (the research session II*: Look for other paths in order not to pay any fine, please. M1: ..... How much fine can I pay?* the most important part is "not to pay" meaning a need *to conclude the work*. For the researcher it was a way of expressing the question "Do you already have all the paths?" indirectly. However, for the boy the word "fine" opened only one possibility of *continuing the work*.

4. Different understanding of the key word. In the research session I (*K02: From the mathematical point of view)* the word 'mathematically' had different meanings for

research and child. For the researcher this word was understood as a concept (relation) from mathematical world, but for the child this word was process (mathematisation) into mathematical world. In the research session II (... *If you take the same path twice, you will pay a fine. Find all the paths without paying the fine)* the word 'fine' used by the researcher had different meanings for both participants. The researcher was aware of the fact that in mathematics listing all the elements of a set does not rule out the situation when some elements repeat. To avoid the situation in which a student counts the same path twice, she established the condition about paying the fine for taking the same path twice. It is highly probable that the student did not understand this information according to the researcher's intention. In his understanding, this word would mean: work carefully, look out, analyse everything what you think is important.

GENERALIZING PHENOMENA CONCERNING OBSTACLES

Below we present opinion, why those phenomena are taking as serious obstacles in mathematical discourse.

1. Different understanding concerning context of situation. In constructivist approach to teaching mathematics mathematical concepts are built by conceptualising the real world. Taking into account this fact mathematical discourse very often takes place on the boundary between real world and mathematical (abstract) world. At the same time so many relations exist between these two worlds. Creating mathematical structure some pieces of knowledge creates new relations and old relation are changed (Hejný, 2004). When a teacher is planning a task or experiment, which can draw pupils close to some mathematical concept, the teacher recognizes this concept in the whole situation. He/she knows what is important and what is not. Pupils who are trying to understand the situation can take into account these phenomena which are close to his/her experience. While solving the problem they could use their own knowledge which, in their opinion, is useful for this situation.

2. Focus on own aims. This obstacle is usually caused by participant's great interest in some particular idea of discourse and it causes that the participant is not opened to the other ideas emerging during the discourse. One participant, occupied by own aims, is very rigid in at least two aspects: he/she sends the information only in one way and interprets the information sent to him/her in the light of own way of thinking. He/she is not able to accept any external information, nor to send any own information in other ways better acceptable by another participant. If this situation lasts long time then the partner could change his/her intension in understanding the situation and starts to defend via own aim. Finally it could lead to the situation that two participants establish their own separate aims. The teacher knows what he aims at when suggesting work on a certain mathematical problem to the pupil. This can be exercising a skill or discovering certain properties and relations, an important form the mathematical point of view. These aims often determine the way of work on the task, and not following the expected patterns of work is often understood as a

mistake. A pupil, when taking a task, aims first of all at finding solution in a possible easiest way, using well-known (or considered most effective) ways of work.

3. Different importance assigned to words in statement. A mathematical task brings different content, depending on which part the receiver focuses. The meaning of the whole statement is understood as we keep in mind the meaning of previously spoken words; the meaning of separate words is put into the broader context and we understand the relations between the parts of the statement. In conveying information each of participants can choose different words or parts of information as most important. In general, one participant conveying information to another participant (in discourse) gives to know what important word is in the information (for example by giving the word at the beginning of sentence or by giving stress on the word). Therefore it is very often not difficult to guess what important word is in the information. But it is not always obvious. Focus on the chosen part of information drives thinking processes in a certain way. If participants focus on different parts then their thinking process is driven differently and consequently different actions emerge. Participants are not usually aware that their thinking is occupied by different ideas and it causes that they divagate from each other.

4. Different understanding of the key word. Difficulty is that the important word is understood differently. Each participant has a different image under one word. Constructing own mathematics is a long-lasting process, during which the meaning of some words can change. Mathematical notions, terms, operations and procedures have at least two faces: they could be understood as concepts or as processes. Different meaning given to the same worlds stimulates taking different actions connected with this particular concept; results of one world evoke different properties, different connections and relations.

Because mathematics is perceived as an exact science it is not thought that different understanding of one word could occur in the mathematical discourse. Mathematics as the school subject could be communicated by at least two ways: in a written way (books, textbooks, etc.) and in discourse. The written text is usually organized in a logical way; one particular word is clearly connected with others, so its meaning is almost clear, usually for more readers at the same level of their education. In the discourse the situation is different: the discourse is spontaneous.

RELATING OBSTACLES TO DESCRIPTIONS IN LITERATURE

In Vygotsky's research, primarily concerned with language and speech, we can find some fragments which implicitly point out the possibility of occurring obstacles, as has also been indicated by us. Vygotsky, showing the connection between thought and language, outlined some key aspects of his theory: relationships between words, and others psychological acts and social phenomena. He also has showed their importance for the proper interpretation of the given statement. Thus Vygotsky, despite not having formulated any categories that deal with obstacles in "teacher –

pupil" communication, drew attention to various phenomena that accompany communication. Some of his remarks are complementary to our conclusions.

1. Different understanding concerning context of situation. Many researchers stress significance of context and individual reception of the whole situation, where the discussion takes a place. Vygotsky precises own point of view, differentiates *sens of word* and *meaning*: The sense of a word is the sum of all the psychological events aroused in our consciousness by the word. It is a dynamic, fluid, complex whole, which has several zones of unequal stability. Meaning is only one of the zones of sense, the most stable and precise zone. A word acquires its sense from the context in which it appears; in different contexts, it changes its sense. In other place Vygotsky says: A word in a context means both more and less than the same word in isolation: more, because it acquires new content; less, because its meaning is limited and narrowed by the context (p.388).

2. Focus on own aims. Vygotsky highlighted, that the aim of utterance shape the whole sentence. Some of his thoughts are as follows: Every sentence that we say in real life has some kind of subtext, a thought hidden behind it (p.399). Behind every thought there is an affective-volitional tendency, which holds the answer to the last "why" in the analysis of thinking. A true and full understanding of another's thought is possible only when we understand its affective-volitional basis (p.402). To understand another's speech, it is not sufficient to understand his words – we must understand his thought. But even that is not enough – we must also know its motivation. No psychological analysis of an utterance is complete until that plane is reached (p. 403).

3. Different importance assigned to words in statement. Focussing on different parts of sentences Vygotsky analysed by using formulation „the lack of coincidence between grammatical and psychological subject and predicate" (p.335). Any part of a sentence may become the psychological predicate, the carrier of topical emphasis: on the other hand, entirely different meanings may lie hidden behind one grammatical structure. Accord between syntactical and psychological organisation is not as prevalent as we tend to assume – rather, it is a requirement that is seldom met. (p.336)

4. Different understanding of the key word. This result has a special importance in Vygotsky's research about the relation between thoughts and words. Vygotsky states: The change of meanings of words and their development – this is our main discovery (p.320). In other place Vygotsky underlines that: The discovery that word meanings evolve leads the study of thought and speech out of a blind alley. Word meanings are dynamic rather than static formations. They change as the child develops; they change also with the various ways in which thought functions. (p.329). Vygotsky continues his idea: each stage in the development of word meaning has its own particular relationship between thought and speech. (p.330).

CONCLUSION

The constructivist approach to teaching mathematics requires some knowledge from a general theory of communication. In mathematical discourse, well-known obstacles in communication have a special meaning. For this reason, it is not so easy to recognise and deal with them. However, this competence could be developed by personal experience with analyses of misunderstandings. During our research, our sensitivity to misunderstandings in mathematics communication has increased. We learned how many and how different obstacles could appear in mathematical discourse. The competence to develop discourse and recognize obstacles in communications should be built on the knowledge from the domains of mathematics, psychology, pedagogy, etc. and on connections among them.

REFERENCES

Bussi, M.B. (1998). Verbal Interaction in the Mathematics Classroom: a Vygotskian Analysis. In H. Steinbring et al. (Eds.), *Language and Communication in the Mathematics Classroom*, The National Council of Teachers of Mathematics, Inc. Reston, Virginia.

Evans, J. (2004). Methods and findings in research on affect and emotions in mathematics education. In Mariotti, M.A. (ed.). *Proceedings of CERME 3 (Third conference of the European Research in Mathematics Education)* [CD-ROM]. Universita di Pisa.

Hejný, M. (1992). Analysis of students' solutions of equations $x^2 = a^2$ and $x^2 - a^2 = 0$. ADUC, n. 1, pp. 65-82.

Hejný, M. (2004). Understanding and Structure. In Mariotti, M.A. (ed.). *Proceedings of CERME 3 (Third conference of the European Research in Mathematics Education)* [CD-ROM]. Universita di Pisa.

Kieran, C. Forman, E. Sfard, A. (2001). Bridging the Individual and the Social: Discursive Approach to Research in Mathematics Education, *Educational Studies in Mathematics, A PME Special Issue,* Vol.46, No.1-3.

Kratochvílová, J., Swoboda, E. (2004). Aspects affecting pupil`s thinking in mathematics during interaction researcher - pupil. In *Proceedings of CERME 3 (Third conference of the European Research in Mathematics Education)* [CD-ROM]. Universita di Pisa.

Mckay M., Davis M., Flanning, P. (2004). *Sztuka skutecznego porozumiewania się*, Gdańskie Wydawnictwo Psychologiczne, Gdańsk 16 – 19.

Nelmes, P. (2004). Developing a conceptual framework for the role of emotions in the language of teaching and learning. In Mariotti, M.A. (ed.). *Proceedings of CERME 3 (Third conference of the European Research in Mathematics Education)* [CD-ROM]. Universita di Pisa.

Polya, G. (1966). Mathematical Discovery. John Wiley & sons, inc., USA.

Steinbring, H. (2005). *The construction of New Mathematical Knowledge in Classroom Interactions, an Epistemological Perspective.* Mathematics Education Library. Springer. Volume 38.

Wygotski, L.S. (1989), *Myślenie i mowa, (Thought and language)* PWN, Warszawa.

WRITING MATHEMATICS THROUGH DOMINANT DISCOURSES: THE CASE OF A GREEK SCHOOL MATHEMATICS MAGAZINE

Anastasia G. Stamou and Anna Chronaki

University of Western Macedonia and University of Thessaly, Greece

In the present article, we examine the use of 'traditional' and 'progressive' discourses in a Greek school mathematics magazine (Euclid A). The analysis indicates that Euclid A seems to draw on both traditional and progressive discourses in order to write mathematics for his/her readers. However, the prevailing lexico-grammatical features are mostly connected with traditional rather than progressive discourse. This means that the general background against which the two discourses are articulated in the textual corpus is that of traditional discourse. From a critical discourse analysis perspective, the 'progressive' discourse enacted in Euclid A functions ideologically: far from ensuring equitable access to school mathematics, it creates confusion. Thus, being subjugated by the dominant traditional discourse, 'progressive' discourse perpetuates the established order in school mathematics.

DOMINANT DISCOURSES IN MATHEMATICS EDUCATION

The traditional/progressive dipole is a cherished one in mathematics education, which reflects two *contrasting*, and at the same time *dominant*, conceptions or 'myths' (Dowling, 1998) about the way school mathematics is (or *is not*) organized around issues of 'classification' and 'framing' (Bernstein, 1990). According to the 'traditional' view, mathematical knowledge transmitted in school needs to follow the logic of academic mathematics (strong classification), while teacher constitutes the 'guardian of science' (Hanrahan, 2006) and possesses absolute control over his/her communication with students (strong framing). According to the 'progressive' view, school mathematics must be connected with the life experience of students (weak classification), while the role of teacher is to facilitate the learning process, and therefore students are given selections for knowledge reception (weak framing).

Traditional discourse has been strongly criticized for limiting the access of students, and especially those from working and lower social class backgrounds, to mathematics, treating science as 'a special truth that only the superintelligent few can understand' (Lemke, 1990, p. 149). Hence, diverse alternative projects have been developed in an attempt to renew the school mathematics curriculum in many countries, such as the 'Realistic Mathematics Curriculum' developed in IOWO, Netherlands (Treffers, 1987), the movement of 'ethnomathematics' in Africa and Brazil (Gerdes, 1996) and the 'Common Sense Activities' of the Genova group in

Italy (Bussi, 1996). Far from being homogeneous, these projects could be broadly characterized as adopting a 'progressive' perspective (for a further discussion see Chronaki, 2000).

Nevertheless, research on mathematical texts, especially from a sociological approach (e.g. Apple, 2000; Dowling, 1998), points out that mathematical texts, whether they draw on a 'traditional' or 'progressive' discourse, they reproduce an ideal cosmos. In a similar vein, Walkerdine (1998) comments that 'traditional' and 'progressive' are two 'common-sense' categories that do not really exist but are in fact socially constructed and serve the politics of the time. Consequently, it seems more important to pay attention on the type of use one makes of 'traditional' and 'progressive' than on the labels themselves.

In the present article, we examine the use of 'traditional' and 'progressive' discourses in a Greek school mathematics magazine (*Euclid A*). In the present context, 'traditional' and 'progressive' constructs are not employed in the sense of evaluating mathematical texts as 'good' (using progressive discourse) or 'bad' (using traditional discourse), but as analytic units for exploring contrasting ways of representing mathematics. The adoption of a Critical Discourse Analysis perspective allows a study of whether and to what extent traditional discourse naturalizes the 'science' of mathematics and subjugates progressive discourse. The analysis of the two discourses is made with the use of the analytic framework of Systemic Functional Grammar of Halliday (1994) and is situated within the context of similar studies conducted in mathematics (Morgan, 1998) and science education texts (Dimopoulos et al., 2005; Hanrahan, 2006; Knain, 2001).

EUCLID A: A SCHOOL MATHEMATICS MAGAZINE

Euclid A is an official magazine of the Hellenic Mathematical Society (HMS) and aims to familiarise students in late primary and early secondary school with mathematics. Its editorial board mostly comprises experienced secondary maths teachers who are, at the same time, authors of articles appearing in the magazine. Although, one witnesses changes in the editorial board over the years, a small number of people consists a stable core. The status of *Euclid A* as a school mathematics magazine is constructed around the fact that a) it is distributed through the HMS at a very low price in almost all schools of the country (and thus it becomes visible and available), and b) its content aims not only to support and extent school mathematics but also to train the gifted ones for entry in the maths Olympiads. The content of text is structured around areas such as theoretical extensions of mathematical concepts (e.g. symmetry, multiplication, polygons), activities (e.g. problems to solve, open problems, exercises), real life problems, general interest and the students' own voice, where they could send their answers to specific problems.

CRITICAL DISCOURSE ANALYSIS

Critical Discourse Analysis (CDA; e.g. Fairclough, 1992; Fairclough & Wodak, 1997) is a sociolinguistic/sociosemiotic approach that acknowledges the central role that language/semiosis occupies in the social life of late modernity (Chouliaraki & Fairclough, 1999). In particular, CDA attempts to bridge linguistic with social analyses of texts, focusing on both the linguistic features of texts and the social structures underpinning those texts. The combination of linguistic with social analysis is achieved through a focus on discursive practice (e.g. the formation of discourse), which is seen as the intersection of linguistic (i.e. text) and social processes (i.e. social practice). Texts are seen as the sites where elements of contrasting discursive practices struggle for dominance over each other. This 'movement' leads to various mixtures of discursive practices within texts, called 'interdiscursivity' (Fairclough, 1992).

Due to its Western Marxist origins, CDA supports the view that language is dialectically related to society. It also puts an emphasis on the ideological role of discourse in perpetuating and legitimizing the dominant representations of the world, through which relations of power are maintained. In particular, CDA is 'critical' in the sense that it unveils naturalized and 'common sense' versions of the world that support the status quo as well as 'hegemonic' worldviews that subjugate and appropriate any alternative to the established order representations of reality.

SYSTEMIC FUNCTIONAL GRAMMAR

Systemic Functional Grammar (Halliday, 1994) is a lexico-grammatical theory which sees language as a network of options from which language users make selections that are ideologically significant. Moreover, language is regarded as being multi-functional, namely, as performing simultaneously an 'ideational' (talk about a specific subject matter), an 'interpersonal' (interact with the hearer/ reader) and a 'textual' (construct the medium of communication) function. Halliday has developed a toolkit for the analysis of these three functions of language, connecting them with specific lexico-grammatical features. For the purposes of the analysis presented below, here we focus in the description of the tools of *vocabulary* and *transitivity* (ideational function), as well as of *personal deixis* and *speech acts* (interpersonal function).

Analysis of ideational function

Language users construct images of social reality by naming it through vocabulary as well as by defining it in terms of causality (i.e. 'who does what to whom'). Causality is expressed via the system of 'transitivity', namely through the determination of

'processes' and 'participants' (Halliday, 1994). In order to determine the nature of processes, Halliday distinguishes among 'material' (doing), 'mental' (sensing), 'relational' (being) and 'verbal' (saying) processes, based on the way processes are worded [1]. Relational processes are further distinguished into 'attributive' (e.g. 'the exercise is difficult') and 'identifying' (e.g. 'the triangle is isosceles'). For the determination of causality, namely, for the determination of the way participants are linked to processes, a distinction is drawn between participants that initiate processes, the 'agents' ('who does') and participants that receive processes, 'the affected' ('to whom something is done').

Analysis of interpersonal function

'Personal deixis' involves all allusions made in a text to the writer and/or reader, which is mainly expressed through the selection of person (i.e. 1st, 2nd and 3rd person of singular or plural number) in personal (e.g. I, we, you) and possessive pronouns (e.g. my, our, your) (Fowler, 1991).

'Speech acts' give a view of language as a tool through which language user expresses his/her intentions and acts (Austin, 1962; Searle, 1969). There are five types of speech acts performed through language. In our data, we found two of them: 'assertive', through which speakers/writers express their belief towards a state of affairs (state, inform) and 'directive', through which speakers/writers ask addressee to do something (request, recommend). Assertive speech acts are oriented to the writer/speaker himself/herself, while the role of the reader/hearer is that of someone being told. Conversely, directive acts are highly interactional in character, being oriented to the reader/hearer, whose role is that of someone being asked for something. For the purposes of the present analysis, four major types of directives were distinguished, according to the degree of power held by the participants of interaction: 'requests' (e.g. 'Find the factor and the main part of the following mononomials') and 'instructions' (e.g. 'In each rectangle parallelogram, we draw a diagonal and we observe that it divides the shape in two equal rectangle triangles'), which assume authority on the part of writer, 'questions', which entail less power on the part of writer (e.g. 'How do we find this number?'), and 'suggestions', which assume relevant equality between writer and reader (e.g. 'Thus, we can use various simple geometrical shapes').

The sociolinguistic profile of traditional and progressive discourses

Drawing upon the work of Morgan (1998) on mathematical texts and upon the studies of Dimopoulos et al. (2005), Hanrahan (2006) and Knain (2001) about science education texts, we sketched the sociolinguistic profile of traditional and progressive discourses, and thus we were able to identify the two discourses in the articles of *Euclid A*.

Specifically, traditional discourse, as a recontextualized discourse of science, has many of the characteristics of dominant academic discourse, which promotes a

positivist view of the world and experience, such as the elimination of any external (to the described world, e.g. human) agency (mathematics as initiator of processes), the adoption of an impersonal style with no interaction between writer and reader (absence of personal deixis), the formulation of definitions and classifications (technical vocabulary, relational identifying processes), and a focus on the transmission of knowledge (assertive speech acts). Moreover, the teacher as transmitter of scientific knowledge has control over the pedagogic process in relation to students (teacher as agent of material processes, second person singular and plural, directive acts of requesting and instructing).

In its attempt to be more familiar and accessible to students, progressive discourse draws on sociolinguistic features from their primary discourse (non-technical and colloquial vocabulary, material processes). An emphasis is also put on human agency and subjective description (humans as agent of material, mental and verbal processes, relational attributive processes, first person of singular number), as well as on the negotiation of mathematics with students (third person of singular and plural number, first person of plural number, directive acts of suggesting).

WRITING MATHEMATICS IN *EUCLID A*

The results from the analysis of the whole textual corpus (Stamou & Chronaki, in preparation) suggest that the texts are interdiscursive, exhibiting features that refer to both discourses. However, the prevailing lexico-grammatical features are mostly connected with traditional rather than progressive discourse. This means that the general background against which the two discourses are articulated in the textual corpus is that of traditional discourse. This has important consequences for the way progressive discourse is textually enacted.

In order to illustrate the way the discourses are interwoven together in the texts, we focus on the analysis of two largely heterogeneous texts but which have a different orientation.Text A mainly draws on traditional discourse, whereas text B has a more progressive perspective. Because articles were four to five pages long, we decided to present some representative extracts of each article in terms of the different styles they employ. We kept all typographic (e.g. bold) and punctuation (e.g. full stops) conventions of the originals.

Analysis of Text A

In this text, three distinct styles were drawn upon, which are clearly distinguished from each other with headings. The text begins with Style 1 (for illustration, see extract 1), which represents 20% of the text's length, followed by Style 2 (see extract 2), which occupies 35% of the text. Finally, Style 3 (see extract 3) is drawn upon, representing 45% of the text's length.

In terms of transitivity, in Style 1, both relational identifying processes initiated by mathematics as well as mental and material processes performed by human agency are employed. Conversely, in Styles 2 and 3, there are only material processes performed by human agents. This stress on human agency has different premises. In Style 2, it is used to describe the work of famous mathematicians, whereas, in Style 3, in order to describe the work of students (reader) in a pedagogic context. Regarding vocabulary, Styles 1 and 3 are very technical, containing many terms from mathematics. The writers seem to be preoccupied with introducing the specialized vocabulary to readers by using bold fonts (this is a practice used extensively throughout the article). It is noteworthy that even Style 2, which does not contain technical words, is treated like such, with the use of bold fonts for the names of famous mathematicians. In interpersonal meanings, Style 1 and 3 are interactional, whereas Style 2 is completely impersonal. Specifically, Style 1 employs the first person plural 'we' to refer to both writers and reader. The speech acts performed are most of them directive and there are few assertive. Directive acts are instructions. Style 2 has no reference to personal deixis, while the speech acts performed are assertive. Style 3 is a traditional pedagogic one: the interaction is constructed on the basis of authority on the part of writers over reader, with the use of the second person plural 'you' to refer to reader and the performance of directive acts of requesting [2].

In conclusion, the text uses three distinct styles, but these do not make the text look contradictory. Each style has a specific place in the text and all of them gather lexico-grammatical features that mainly relate to traditional discourse. There are some exceptions, which are not, however, disconnected from the whole. Although mental processes are typical characteristics of progressive discourse, in the present context, they are rather linked to traditional one. In particular, the specific mental process used (i.e. 'suppose' in contrast to the pedagogic mental 'observe' or 'see') is commonly used in academic mathematics, in which case, the reader is addressed as a thinker and invited to join the authors and institute a common world (Morgan, 1998). On the other hand, because of the specific context, the first person plural 'we' employed is rather ambiguous in its interpretation: it could be construed as a reader-inclusive pedagogic 'we' (progressive discourse), but also as a reader-exclusive academic 'we' (traditional discourse).

Text A

Style 1 'A problem'

…We can suppose that the fraction a/b is **improper** (i.e. that its terms have no other common divisor than unit), because if it is not improper we can always make it improper by simplification. We suppose that the numbers a and b are prime between them (i.e. G.C.M. (a, b)= 1 and G.C.M. $(3^2, 5^2)$= 1, since a common divisor of 3^2= 9 and 5^2= 25 must be common divisor also of 3 and 5)…We concluded to **'reductio ad absurdum'** because we accepted that the equation (1) has solution **a rational number**. Thus, we conclude that such

a rational number does not exist. In other words, the measure of hypotenuse of rectangular triangle ABC is not a rational number.

Style 2 'From the history of irrational numbers'

Irrational numbers were discovered by the School that the philosopher **Pythagoras** from the island of Samos established in Krotonas of South Italy in the 6th century B.C. It is told that the student of Pythagoras, **Ippassos**, discovered them…The asymmetric magnitudes became known, and the great mathematician of antiquity **Eudoxos** made a theory that founds the ratios and the analogies between any similar magnitudes (symmetric or asymmetric). This theory was included in **'Elements of Euclid'**.

Style 3 'Exercises that we propose'

- Find the numerical value of algebraic representation:

$5z^4y^4o^5/2x^3$, if x= 1, y= -1, o= 2, z= -2

- Find the final form (i.e. the form that results from operations and reductions of same terms) of the following representation:

(x-1) (x-2) + (x-2) (x-3) + (x-2) (x-8)

- Determine the l, so that the polynomial: x^3+2x+l, be perfectly divided with (x-2).

Analysis of Text B

In this text, four distinct styles were drawn upon. Styles were not distinguished from each other in a consistent way [3]. The text begins with an alternation between Style 1 (23% the text's length) and 2 (31% of the text). Then, Style 3 is drawn upon (24% of the text). Next, there is a frame, in which Style 2 is again employed for a while, and finally Style 4 is drawn upon (22% of the text), signaled by a heading.

In terms of transitivity, all extracts refer to human agency. In Style 1, human agency stands for the writer, who mainly initiates mental processes in order to describe his personal perceptions of mathematics. In Style 2, human agency is represented by a famous mathematician, who performs material processes for the description of his work. In Style 3, the human agent is generic (i.e. writer, reader, whoever), who initiates material processes for the making of calculations in an unspecified context (it could be pedagogic, domestic etc.). In Style 4, human agency is represented by the reader-student, who performs material processes for the description of mathematical operations in a pedagogic context. Regarding vocabulary, Style 2, 3 and 4 involve semi-technical words, whereas Style 1 contains colloquial lexis. In interpersonal meanings, Styles 1, 3 and 4 are interactional, whereas Style 2 is completely impersonal. Specifically, Style 1 is written in first person singular by referring to writer, and thus it adopts a highly personal style. The speech acts performed are assertive, which is linked to the narrative character of the Style. Style 2 does not make any reference to personal deixis, while the speech acts performed are assertive. Style 3 constitutes a mixture of traditional and progressive pedagogic style: the writer exerts his power over his reader by performing directive

acts of instructing. However, the use of the third person singular 'one/ he' attenuates his authority, by speaking in generic terms, and thus it is a more negotiable style, echoing progressive discourse. Finally, Style 4 represents a traditional pedagogic style: the interaction is constructed on the basis of authority on the part of writer over reader, with the use of the second person singular 'you' to refer to reader and the performance of directive acts of requesting.

In conclusion, like text A, text B also uses distinct styles. Contrary to text A, though, in which the distinct lexico-grammatical features form a coherent whole, in text B, they seem to create discontinuity. In fact, the writer seems indecisive between traditional and progressive discourse. At the beginning of the text, he alternates between them (Style 1 and Style 2), whereas, next, he adopts a mixture of the two discourses (Style 3), concluding to traditional discourse (Style 4). Moreover, some of the contrasting elements represent 'marginal cases' in respect to the whole textual corpus analyzed (Stamou & Chronaki, in preparation). Thus, the highly personalized Style 1, which refers to progressive discourse, was rarely used in general. On the other hand, the authoritative stance of writer over reader of Style 4, and especially the use of second person singular (and not plural), being part of traditional pedagogic style, was also rarely employed in the corpus. Finally, the intimacy created between writer and reader by Style 1 at the beginning of the text, which strikes the reader, because it is not a common stylistic option in *Euclid A*, is cancelled by the prevailing traditional discourse in the rest of the text (Style 2, Style 4 and Style 3 in part). Furthermore, this highly personalized style seems to be actually exploited for the use of a highly authoritative style (Style 4), which could be easier accepted, because it has gained the reader's trust, having being offered in a 'progressive' wrapping.

Text B

Style 1 Turning over the pages of the school textbook of Informatics, my eyes caught a picture. In its subtitle, I read that it is about Napier rods…It was the first time that I heard of Napier rods. I did not understand much from the picture, and the text did not explain much on them either. How were they and how were they used? Because they drew my interest, I decided to search about them.

Style 2 The machine of Schickard is considered to be the first calculator and was constructed between 1620 and 1623, in an attempt to automate astronomic calculations. For the construction of his machine, Schickard, who was professor in the university, relied on the so-called Napier rods… In 1614, the Scot J. Napier constructed a series of rods with which he could make easily and simply calculations. For example, he could make the most difficult multiplication into a simple addition.

Style 3 If one had to multiply 456 x 2, he should take the rods of 4, 5 and 6 and put them the one next to the other in this order. Next, he should consult the second row in which the multiples of each number (4, 5, 6) were written and should add the numbers he found which were written diagonally…If, on the other hand, he had to find the product 456 x 52, then he should write 52 as a sum of tens and its units, that is, 52 = 50 + 2. Next, he should find in the rods the product 456 x 5 and add up to this a zero (5 tens).

<u>Style 4</u> 'And now it's your turn'

Make rods like these of Napier with paper or with the rods of ice cream and use them to make various multiplications. Execute the multiplication 268 x 34, first with the Napier rods and then with the method of Arabs. Verify the result by executing the multiplication with the manner you know…Place the digits 1,4, 6, 8, 9 in squares and find the highest and the lowest product.

CONCLUDING REMARKS

Euclid A seems to draw on both traditional and progressive discourses in order to write mathematics for his/her readers. However, the prevailing lexico-grammatical features are mostly connected with traditional rather than progressive discourse. This means that the general background against which the two discourses are articulated in the textual corpus is that of traditional discourse. Therefore, even texts that have a more progressive orientation necessarily also draw on features of traditional discourse, resulting in the formation of contradictory and discontinuous texts. From a critical discourse analysis perspective, the 'progressive' discourse enacted in *Euclid A* functions ideologically: far from ensuring equitable access to school mathematics, it creates confusion. Thus, being subjugated by the dominant traditional discourse, 'progressive' discourse perpetuates the established order in school mathematics.

NOTES

1. In the framework of mathematics that we study, processes such as 'add', 'calculate' and 'measure', despite being of intellectual nature, following Morgan (1998), they are treated as material rather as mental processes. Specifically, Morgan maintains that such processes give the impression that mathematics concern 'doing' certain things, namely, manipulating numbers, symbols and shapes. In contrast, mental are processes like 'think', 'conclude' and 'suppose', which give the impression that mathematics concern 'sensing' certain things.

2. The distinction between the second person of singular and that of plural number is grammatically signalled in Greek.

3. The transition from one style to the other was signalled with italics, with frames, with headings or with no sign at all.

REFERENCES

Apple, M.: 2000, *Official Knowledge: Democratic Education in a Conservative Age*, Routledge, London.

Austin, J.: 1962, *How to Do Things with Words*, Oxford University Press, Oxford.

Bernstein, B.: 1990, *Class, Codes and Control, Vol. IV, The Structuring of Pedagogic Discourse*, Routledge, London.

Bussi, M: 1996, 'Mathematical discussion and perspective drawing in primary school', *Educational Studies in Mathematics*, 31, 11-41.

Chouliaraki, L. and Fairclough, N.: 1999, *Discourse in Late Modernity*, Edinburgh University Press, Edinburgh.

Chronaki, A.: 2000, 'Teaching maths through theme-based resources: pedagogic style, "theme" and "maths" in lessons', *Educational Studies in Mathematics*, 42, 141-163.

Dimopoulos, K., Koulaidis, V. and Sklaveniti, S.: 2005, 'Towards a framework of socio-linguistic analysis of science textbooks: the Greek case', *Research in Science Education* 35, 173-195.

Dowling, P.: 1998, *The Sociology of Mathematics Education*, Falmer Press, London.

Fairclough, N.: 1992, *Discourse and Social Change*, Polity Press, Cambridge.

Fairclough, N. and Wodak, R.:1997, 'Critical Discourse Analysis', in T.A. Van Dijk (ed.), *Discourse as Social Interaction*, Sage, London, pp. 258-284.

Fowler, R.: 1991, *Language in the News: Discourse and Ideology in the Press,* Routledge, London.

Gerdes, P.: 1996, 'Ethnomathematics and mathematics education', in A.J. Bishop, K. Clements, C. Keitel, J. Kilpatrick and C. Laborde (eds), *International Handbook of Mathematics Education*, Kluwer Academic Publishers, pp. 909-945.

Halliday, M.A.K.: 1994, *An Introduction to Functional Grammar, 2nd edition*, Edward Arnold, London.

Hanrahan, M.U.: 2006, 'Highlighting hybridity: a Critical Discourse Analysis of teacher talk in science classrooms, *Science Education*, 90, 8-43.

Knain, E.: 2001, 'Ideologies in school science textbooks', *Science Education*, 23(3), 319-329.

Lemke, J.L.: 1990, *Talking Science: Language, Learning and Values*. Ablex, Norwood, NJ.

Morgan, C.: 1998, *Writing Mathematically: The Discourse of Investigation*, Falmer Press, London.

Searle, J.: 1969, *Speech Acts*, Cambridge University Press, Cambridge.

Stamou, A.G. and Chronaki, A. (in preparation) 'Doing mathematics "traditionally" and "progressively": the case of a Greek school mathematics magazine'.

Treffers, A.: 1987, *Three Dimensions: A Model Goal and Theory Description in Mathematics Instruction – The Wiskobas Project*, Mathematics Education Library, D. Reitel Publishing Company.

Walkerdine, V.: 1998, *Counting Girls Out: Girls and Mathematics*, Falmer Press, London.

INVESTIGATING THE INFLUENCE OF SOCIAL AND SOCIOMATHEMATICAL NORMS IN COLLABORATIVE PROBLEM SOLVING

Konstantinos Tatsis

University of the Aegean, Greece

The interactions of students while they cooperate to solve a mathematical problem are based on a number of shared assumptions, i.e. norms. These norms may be categorised into social and sociomathematical. We investigate how these norms are established during joint problem solving sessions of university students. Our results, in line with relevant studies, show that students jointly establish a set of assumptions concerning their behaviour; moreover, certain norms are related to particular aspects of the problems posed. Most of these norms enhance the problem solving process. However, exceptions do exist, but they have a local orientation and a relatively small influence.

INTRODUCTION

The interactions contained in mathematics teaching are a rich field for analysis; contemporary researchers draw their attention to various aspects of them in order to clarify issues related to mathematical learning. Certain studies focus on the way mathematical meanings are interactively created and established; this is usually done by a linguistic content analysis of the discussions. Sfard (2001) for example, uses a network flowchart to demonstrate how meanings are gradually constructed by the speakers. Pimm (1987) examines the connection between mathematical and everyday language; similar is the work of Pirie (1998) and Moschkovich (2003). Other studies incorporate the social-psychological factor in their analyses by examining the various rules (and meta-rules) that regulate the interaction (Sfard, 2000; Yackel and Cobb, 1996; Yackel, 2001) or by interpreting the participants' acts from a certain social or psychological perspective (Mercer, 1995; Rowland, 2000). Most of these studies examine teacher-student interactions with the exception of Yackel (2001) and a small part of Rowland (2000). The presence of the teacher has serious effects on the flow of the interactions, since his/her authority influences the whole process. This fact has led us to perform our research with no teacher (or researcher) intervention; another important aspect of our study was the inexistent experience of the Greek students in joint problem solving. Thus, we were interested to see how the students would coordinate their acts in order to successfully deal with the problems. Particularly, we were interested in identifying the norms that regulated the process and studying their effect on it. Moreover, we aimed at relating these norms with Greek students' views concerning mathematics.

THEORETICAL FRAMEWORK

The most influential theory that studies human interactions is symbolic interactionism (Blumer, 1969; Mead, 1934), which is based on the assumption that human

interactions are made possible through the manipulation of various linguistic and non-linguistic symbols. When a person enters a particular setting (e.g. a room where s/he is about to solve a mathematical problem) firstly s/he defines the situation in which s/he is involved; this is done by using existing information about the setting, the participants and the possible (or desired) outcomes of the interaction. For example, when a student enters a problem solving session, s/he might have some information concerning the other participants or whether the session will be audio- or video-taped, etc.

While the interaction proceeds, all participants jointly reach a definition of the situation, i.e. a mutual agreement as to whose claims concerning what issues will be temporarily honoured (Goffman, 1971, p. 21). This definition is continuously renegotiated as various prescriptions take effect. The nature and the function of these prescriptions has been the focus of a number of sociological studies. Biddle and Thomas (1966) define prescriptions as "behaviours that indicate that other behaviours should (or ought to) be engaged in. Prescriptions may be specified further as demands or norms, depending upon whether they are overt or covert, respectively" (p. 103). For example, each culture possesses certain rules concerning behaviour in social interactions, from the clothes one is expected to wear to the way one addresses the other. Norms, on the other hand, are covert in nature; Homans (1966) describes them as ideas which "can be put in the form of a statement specifying what the members [of a group] or other men should do, ought to do, are expected to do, under given circumstances" (p.134). For example, a student is not supposed to interrupt the teacher's talk; on the contrary, the teacher may interrupt the student in order to correct him/her or to maintain the class order. While doing mathematics, students and teachers adhere to similar rules and norms; generally, they adhere to general rules and norms that apply to every social interaction, like the norm concerning the interruptions we have already mentioned. Moreover, participants adhere to a set of sociomathematical norms, i.e. "normative aspects of mathematics discussions specific to students' mathematical activity" (Yackel and Cobb, 1996, p. 461). An example of a sociomathematical norm is the understanding of what counts as mathematically different, sophisticated, efficient and elegant (Yackel, 2001). The question if similar norms would appear in a different cultural and educational setting was the main reason for conducting the present study.

METHODOLOGY

The participants of our study were 40 undergraduate students of the Department of Primary Education (i.e. pre-service teachers) of the University of Ioannina in Greece. The students participated voluntarily in the study. They were asked to choose their partner in order to form 20 pairs. Thirteen pairs consisted of two females, six pairs consisted of a female and a male and one pair consisted of two males. Three one-hour sessions were held for each pair and one problem was assigned at a session. The sessions were held in a laboratory setting with only the two students and the observer present. We chose this setting for our convenience concerning the analysis of the

dialogues; all students were familiar with the particular room, since it was used for many purposes. The only instructions given to the students were that they should verbalize every thought they make and that they should try to cooperate to solve the problems posed. The students were aware that the sessions were tape-recorded by the observer, whose interventions were the fewest possible (e.g. he did not reply to questions like "Is this the right thing to do?"). The time interval between the sessions of each pair varied between four to seven days.

Once the dialogues were transcribed into written text we performed a two-level analysis (Lemke, 1989; Mercer, 1995). In the first level – the thematic analysis – we looked at the way mathematical concepts were created and negotiated during the interactions. The process of thematic system development (Lemke, 1989) consists of patterns of talk found in the text; these patterns include adjacency pairs (i.e. two turns where the first establishes the 'conditional relevance' of the second, like question-answer) or larger conversational units. Our intention was to trace the mathematical concepts and procedures from the moment they were introduced to the moment they were used in the solution process.

In the second level – the interactional analysis – which is the main focus of this paper, we initially looked at the way language was used by the participants to convey attitudes related to particular social and sociomathematical norms. Norms were "inferred by identifying regularities in patterns of social interaction" (Yackel and Cobb, 1996, p. 460). Then, the effect of these norms was examined by analysing the thematic patterns in each case. The usual pattern one expects to find is introduction-discussion-approval/disapproval; whenever there were changes or significant delays in that pattern, we looked for connections with particular norms that were established. Thus, the two levels of analysis were not in fact separated; what we did was observe how the norms established influenced the thematic patterns. The analysis section that follows, will clarify this procedure.

SAMPLE ANALYSIS

Two examples will be presented, in order to demonstrate the analytic procedure. All discussions are translated from Greek by the author.

Paula and Joanna

Paula and Joanna are the pseudonyms of two 22-year-old female students. Like all the pairs in the study they have known each other prior to the study. The following excerpt comes from their first session, when they were assigned the 'T-shirt problem':

> The design below is going to be used on a T-shirt. You accidentally took the original design home, and your friend, Chris, needs it tonight. Chris has no fax machine, but has a 10 by 10 grid just like yours. You must call Chris on the telephone and tell him precisely how to draw the design on his grid. Prepare for the phone call by writing out your directions clearly, ready to read over the telephone.
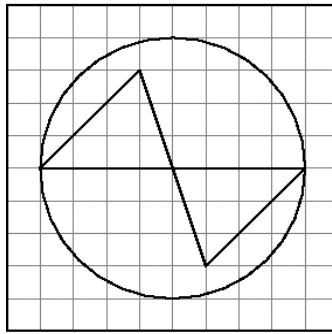
In the preceding part of the dialogue (1-34), the two students have agreed on the instructions concerning the drawing of the circle and its diameter.

35	Joanna	These are a problem. [She refers to the triangles in the figure] He should draw a straight line again…
36	Paula	Where from?
37	Joanna	From the point where the diameter touches the circle…
38	Paula	Yeah, but how will he know where from, on what point? Look, if we tell him that from here…
39	Joanna	And?
40	Paula	Tell him… in the square [inaudible] purely practically, describe the squares to him, count the squares and tell him go to the particular square and draw a line…
41	Joanna	You mean to put numbers in the small squares?
42	Paula	Yeah, but this is totally practical, it's not mathematical…

At the beginning of the excerpt, the concept of a straight line is introduced by Joanna; this concept belongs to the broader concept of the triangle included in the figure. To be precise, Joanna refers to a line segment, a concept which is understood by Paula, who identifies a basic property of it, i.e. its starting point. Joanna's reply in 37 is not explicit enough for Paula, who reacts in 38. Her reaction reveals the following norms:

- the social norm that if you disagree with someone's opinion you are expected to justify your view;

- the social norm that in order to achieve a smooth cooperation and to avoid tension you are expected to express your disagreement in an indirect way (this is revealed by the "Yeah" at the beginning of Paula's utterance);

- the social norm of cooperation; the students are expected to work together (this is revealed by the first plural person in "we tell him");

- the sociomathematical norm that a mathematical proposition is expected to be unambiguous;

- the sociomathematical norm that a mathematical method is expected to be understood by a third person who reads it (in our case Chris).

In her next turn (40) Paula articulates her suggestion: she proposes the use of a system of coordinates. All the verbs she uses are in the first plural person, a fact that

goes in line with the 'cooperation norm' we have mentioned. Paula's suggestion reveals the sociomathematical norm that there is a distinction between mathematics and general practice (however this may be conceived by the speakers). In 40 and 42 Paula stresses the fact that her process is "totally practical" and "not mathematical"

The next part of the session (53-137) contains the students' continuous attempts to generate a set of clear instructions for the drawing of the two triangles. They reach a dead end because their expressions lack precision. Then, Joanna re-introduces Paula's suggestion on the use of a coordinates system.

| 138 | Joanna | What do we call these? The ones we… diagrams, not diagrams, what do we call them? |
| 139 | Paula | You mean in mathematics, this… |
| 140 | Joanna | Yeah, the one we drew an horizontal and a… an horizontal and a vertical… the ones we called x and y? |

This excerpt reveals the sociomathematical norm that mathematical terminology is the desired form of expression when solving a mathematical problem; however we believe that this norm is rooted in the "non-ambiguity" norm.

Finally, the two students adopt the use of a coordinates system, but the fact remains that they have wasted a considerable amount of time because of their common attitude towards the distinction between mathematics and practice.

Tania and Sofia

Tania and Sofia are the pseudonyms of two 21-year-old female students. The following excerpt comes from their second session, when they were assigned the 'triangle problem':

> The picture below shows a triangle in which 3 lines are drawn to one or the other of the opposing sides from each of two vertices. This divides the triangle into 16 non-overlapping sections. If 14 lines are drawn in the same way, how many non-overlapping sections will the triangle have?
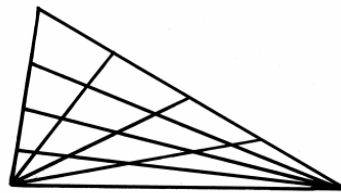
In the beginning of the session (1-30) Tania and Sofia misunderstood the problem; they thought that they were asked to find the number of line segments which produce 14 non-overlapping sections. After resolving this issue, they agreed on the fact that drawing 14 segments is difficult and time-consuming, so they started formulating a joint method based on analogies:

| 77 | Tania | We should do it that way, see how many we get and how else we could do it. |

78	Sofia	Yeah, and then we see, then we'll see it. Let's do it that way. So, shall we do it with stepping on the unit? Yeah, let's do it with stepping on the unit.
79	Tania	How would that help you?
80	Sofia	Come on, let's get it over with!
81	Tania	No, we shan't do anything stupid. What's your rationale? I mean…
82	Sofia	To get a result.

Sofia then, by the use of analogies (3 line segments produce 16 sections, therefore 14 line segments produce x sections) gets the result of 74.66 sections.

88	Sofia	That's the number of sections.
89	Tania	Is this possible? Does this make sense?
90	Sofia	That's the outcome!
91	Tania	I can see that.
92	Sofia	What did you expect?
93	Tania	It doesn't make sense. 74.6 doesn't make sense. It would either be a section or not.
94	Sofia	What can I do? That's the outcome.
95	Tania	So we must do something else. This doesn't make sense.

Unlike the previous example, the theme under discussion here is not a concept and its properties, but a mathematical process. This fact leads to the establishment of a different set of norms related to how a mathematical method may be validated. Tania in 77 introduces the idea that a mathematical method may be validated by its outcome; this idea is immediately accepted (so it becomes a sociomathematical norm) by Sofia, who goes on describing the method she intends to implement. Another matter that arises is the justification of a method (79). Sofia, in response to that, provides a weak justification of the analogies method (82). In this case the participants have different understandings of what constitutes a sufficient justification of a mathematical method. Maybe that explains the tension we see, especially in Sofia's talk (80, 90 and 94). By implementing the analogies method Sofia gets a decimal number of sections; this is the point when another sociomathematical norm is established: the outcome of an analytic procedure (or even of a single operation) is expected to 'fit' in the context of the problem. In our case, we need to have a whole number of sections, a fact stressed by Tania and, finally, sort of acknowledged by Sofia in 94. We may note that the 'cooperation norm' found in the previous pair is also present here: all verbs in 77, 78, 81 and 95 are in the first plural person. But there are certain points when one participant (in our case Sofia) does not adhere to it; this leads to moments of tension. Finally, Tania's utterances reveal her attitude towards expressing her opinion in a non-offensive way (79, 89), a norm we have found in the previous pair too.

The above analysis demonstrates that an understanding of participant (no matter if it becomes shared or not) may hinder or slow down the solution process; if the

collaboration norm is absent, i.e. if the participants do not jointly resolve their disputes, the process becomes problematic.

From the thematic analysis point of view we may note that analogies play an important part in students' thematic repertoire: almost half of the pairs implemented a similar method before proceeding to another solution.

RESULTS

The analysis of the transcribed discussions has led us to the identification of nine norms. The first three are social norms and the remaining six are sociomathematical norms. All these norms have appeared in the sample analysis section.

a) Collaboration norm: the participants are expected to reach a mutual agreement on the solution process and its features. It is expressed through the first plural person of the verbs and the questions about the partner's opinion before implementing a method.

b) Justification norm: one has to justify his/her opinion, especially when s/he expresses disagreement with his/her partner. It is expressed through words such as "because" or "that's why" in a sentence.

c) Avoidance of threat norm: one is expected not to threaten his/her partner, i.e. not to insult him/her. It is expressed through indirect speech acts (Austin, 1962) or by various politeness strategies (Brown and Levinson, 1987).

d) Non-ambiguity norm: mathematical expressions are expected to be clear and unambiguous. It is expressed through prompts for rephrasing, using more "accurate" or strictly mathematical terms.

e) Third person comprehension norm: mathematical expressions are expected to be explicit enough so they can be understood by a third person that reads them. This norm is related to the non-ambiguity norm and is expressed through prompts for rephrasing, enhanced with references to the third person.

f) Mathematical justification norm: mathematical methods need justification before their implementation; a rationale is needed to support their use. This is mainly expressed through questions beginning with "why".

g) Mathematical differentiation norm: mathematical areas such as algebra and geometry are distinct; there is also a differentiation between mathematical and everyday practices. In the 'triangle problem' it takes the form of differentiating between the 'geometrical' solution (i.e. draw 14 line segments) and the 'algebraic' one (i.e. find a formula that gives the number of the sections).

h) Validation norm: mathematical methods need to be validated before or/and after they are implemented. A method may be validated by its difficulty, its time duration or even its result. In some cases the method may be validated by whether it a 'pure mathematical' or a 'practical' one. This norm is expressed through queries for information on the above matters, or through quotes, such

as: "Forget about it, it'll take us ages to do that" (taken from a pair discussing whether they should draw 14 line segments in the 'Triangle problem').

i) Relevance norm: the outcome of a method is expected to be relevant to the problem's conditions; in other words, the result has to make sense. It is related to the validation norm, since an irrelevant result may probably lead to the withdrawal of a method.

From the thematic development point of view, all norms influence the process of establishing a mathematical concept or method. The following diagram demonstrates the points when each norm, once established, may take part in the solution process.
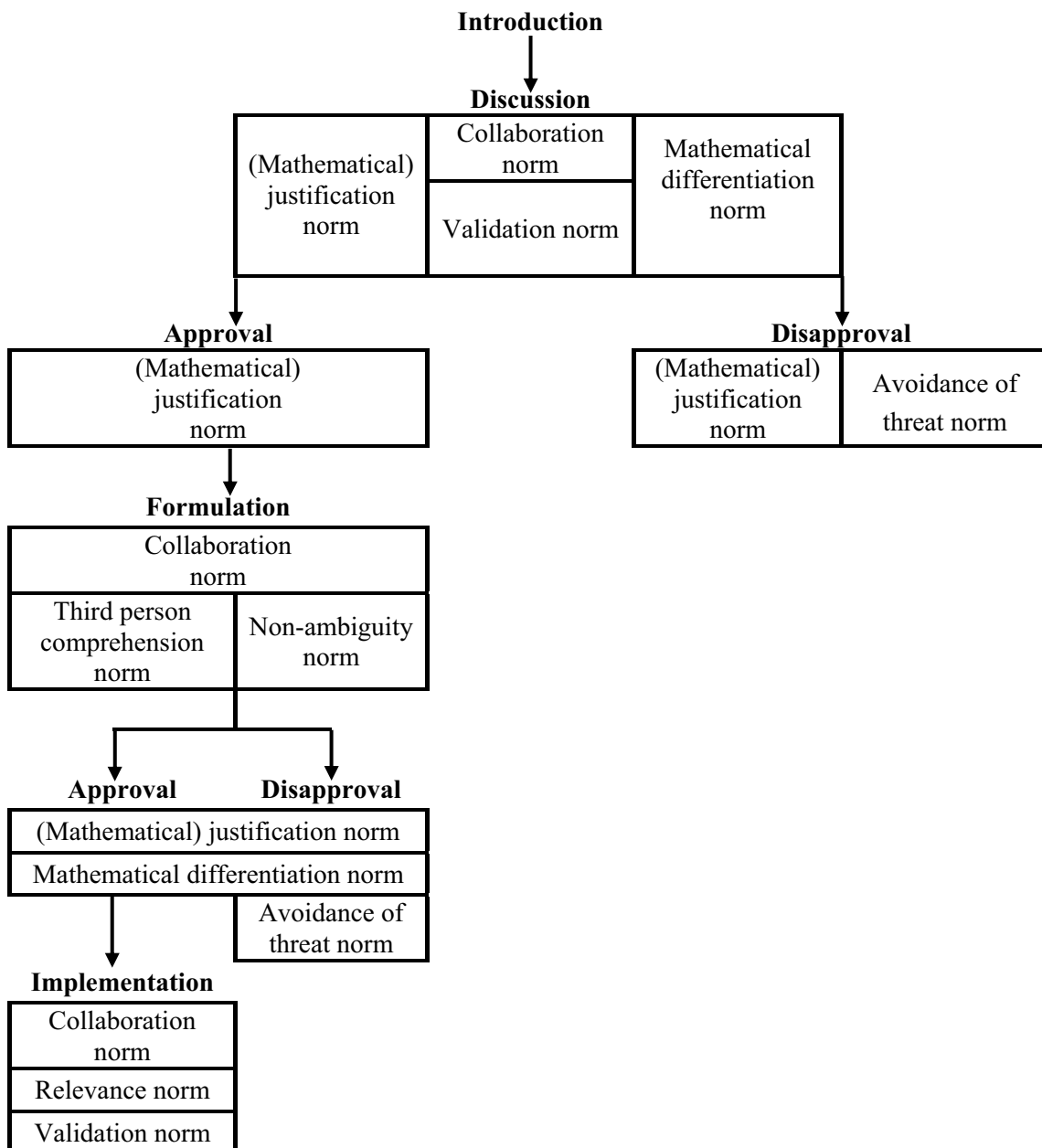
Introduction

Discussion

| (Mathematical) justification norm | Collaboration norm | Mathematical differentiation norm |
| | Validation norm | |

Approval

| (Mathematical) justification norm |

Disapproval

| (Mathematical) justification norm | Avoidance of threat norm |

Formulation

| Collaboration norm | |
| Third person comprehension norm | Non-ambiguity norm |

Approval   **Disapproval**

| (Mathematical) justification norm |
| Mathematical differentiation norm |
| | Avoidance of threat norm |

Implementation

| Collaboration norm |
| Relevance norm |
| Validation norm |

Diagram 1: Norms and thematic development

The influence of most of the above norms has been positive as far as the concept development is concerned; indeed, as the sample dialogues have shown, students have jointly established norms related to smooth cooperation and mathematical justification. The absence of the 'collaboration norm', even in small parts of the discussions, has led to moments of tension and disorder. Problems in the process were also caused by the mathematical differentiation norm; the students were sometimes caught between that norm and the validation norm. In other words, they validated a method not by its difficulty or its efficiency, but by its 'mathematical' or 'non-mathematical' character.

CONCLUSION

Our study was driven by two basic needs: the need to expand the existing research in the field in a different cultural and educational setting and the more general need to further clarify the concept of norm in order to make it a useful tool in mathematics education. Concerning the first need, our results have shown that the norms found in our setting are similar to those identified in relevant studies. However, this "universalism" is not unquestionable: Greek students seem to hold specific views towards mathematics and its usefulness, especially when it comes to solving a problem. It seems that mathematics education in Greece has established some norms which sometimes hinder problem solving. According to these norms, mathematics is comprised of distinct and non-related areas, such as algebra and geometry. Moreover, mathematical language is seen to be comprised mainly of formal terminology, where everyday language has no place.

The previous remarks may serve as a useful tool for the mathematics teacher to work upon. We have shown how social and sociomathematical norms can be identified by particular utterances or patterns of talk and then be related to phases of concept development (see Diagram 1 in the previous section). This scheme may be used by the teacher to identify the established norms (and maybe elaborate or alter them) and organise his/her interventions whenever necessary (i.e. whenever s/he judges that the norm established may hinder the problem solving process). We believe that, handled in this way, norms are a useful tool in mathematics education. Moreover, they can help us to better understand the way people interact, exchange views and create shared mathematical knowledge in various educational or everyday settings.

REFERENCES

Austin, J.: 1962, *How to do things with words,* Harvard University Press, Cambridge, MA.

Biddle, B. J. and Thomas, E. J.: 1966, 'Prescriptions', in B. J. Biddle and E. J. Thomas (eds.), *Role Theory: Concepts and Research,* John Wiley and Sons, Inc., New York, pp. 103-104.

Blumer, H.: 1969, *Symbolic Interactionism: Perspective and Method,* Prentice-Hall, Englewood Cliffs, NJ.

Brown, P. and Levinson, S. C.: 1987, *Politeness: Some universals in language usage,* Cambridge University Press, Cambridge.

Goffman, E.: 1971, *The Presentation of Self in Everyday Life,* Penguin University Books, Harmondsworth, Middlesex.

Homans, G. C.: 1966, 'Norms and Behavior', in B. J. Biddle and E. J. Thomas (eds.), *Role Theory: Concepts and Research,* John Wiley and Sons, Inc., New York, pp. 134-144.

Lemke, J. L.: 1989, *Using language in the classroom,* Oxford University Press, Oxford.

Mead, G. H.: 1934, *Mind, Self and Society,* University of Chicago Press, Chicago.

Mercer, N.: 1995, *The Guided Construction of Knowledge: talk amongst teachers and learners,* Multilingual Matters, Clevedon.

Moschkovich, J.: 2003, 'What counts as mathematical discourse?', in N. A. Pateman, B. J. Dougherty and J. Zilliox (eds.), *Proceedings of the 27th Annual Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, Honolulu, USA, pp. 325-331.

Pimm, D.: 1987. *Speaking mathematically,* Routledge and Kegan Paul, London.

Pirie, S. E. B.: 1998, 'Crossing the Gulf between Thought and Symbol: Language as (Slippery) Stepping-Stones', in H. Steinbring, M. G. Bartolini Bussi and A. Sierpinska (eds.), *Language and Communication in the Mathematics Classroom,* National Council of Teachers of Mathematics, Reston, VA, pp. 7-29.

Rowland, T.: 2000, *The Pragmatics of Mathematics Education: Vagueness in Mathematical Discourse,* Falmer Press, London.

Sfard, A.: 2000, 'On reform movement and the limits of mathematical discourse', *Mathematical Thinking and Learning* 2 (3), 157-189.

Sfard, A.: 2001, 'There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning', *Educational Studies in Mathematics* 46, 13-57.

Yackel, E. and Cobb, P.: 1996, 'Sociomathematical norms, argumentation, and autonomy in mathematics', *Journal for Research in Mathematics Education* 27 (4), 390-408.

Yackel, E.: 2001, 'Explanation, justification and argumentation in mathematics classrooms', in M. van den Heuvel-Panhuizen (ed.), *Proceedings of the 25th Annual Conference of the International Group for the Psychology of Mathematics Education,* Vol. 1, Utrecht, the Netherlands, pp. 9-24.