WORKING GROUP 3
Building structures in mathematical knowledge

CONTENTS

BUILDING STRUCTURES IN MATHEMATICAL KNOWLEDGE

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Fourteen papers were submitted to G3 written by 24 authors. One author submitted two papers and was asked to decide which she wished to present. Hence thirteen papers went through the reviewing process.

The review procedure comprised two stages. All papers were sent to two of the other authors for the initial reviews. These reviews were then submitted to a designated team leader who wrote a covering review, which was sent to the author of the paper.

Three papers were returned to the authors for major amendment and these were not resubmitted. One of the accepted papers cannot be published since the author did not attend the conference. Hence nine papers were discussed during the conference.

The 17 participants who attended all sessions of the group included five who were also at CERME3. Everyone was asked to read all the papers and to prepare questions and comments in writing for each paper. These were then given to the authors in advance with the request that they gave a brief overview of their research but to spend most of the forty minutes allotted to them, responding to the written questions and comments. We found that this provoked excellent discussion and deep probing of the research and analysis given in each paper. The discussion of each paper was chaired by one of the Group Leaders. Notes were taken and these were used as the basis for making any necessary amendments. The papers have been resubmitted and the review procedure re-enacted. They are now published in this section of the proceedings.

It should be noted that the diversity of topics covered in the papers, which were reviewed, was extremely wide, from the point of view of research problem, methodology used and theoretical background. An innovation occurred during one of the presentations which was agreed beforehand by all the Group Leaders. The authors of the paper dealing with ‘mathematical thinking through oral communication’ felt that the best way that the participants could understand their research methods was to run a mini-workshop. This method was well received by the participants and the Group Leaders decided to offer this method of presentation for CERME5. The participants could be offered challenging tasks, which might be solved at or away from the conference or even tried by participants in their own teaching. Two of the papers presented at CERME4 were the results of international collaboration and it is felt that the proposed method of presentation will help to establish additional trans-national research and collaboration.

It was decided to discuss the papers under the following six headings:

Methodology

Methodology was a frequent point of discussion. Two aspects of this discussion are raised below:

If a group of researchers work for several years together, they usually develop their own methodology, technology and language and sometimes are not aware of how difficult it is for a person not familiar with their work to follow their presentation. A set of illustrations appeared to be the best way to overcome this communication gap.

If the presenter of a paper does not put stress on issues which are important for other participants of the group, it causes concern. For example if an experiment is not at least audio-recorded and relies on memory of the researcher or even a third person, then the analysis of such research data must raise doubts about its validity in the eyes of those researchers for whom this is profoundly important.

Phylogeny and Ontogeny

How can the history of mathematics be used for the teaching of mathematics? If a teacher is unaware of the history of mathematics, does this make him/her less able to understand the development of mathematical topics within the mathematical syllabus? For instance one paper showed how the development of the language of algebra extended over centuries whilst in school this is done over a very short period. For example the introduction of the language of letters in history was an organic developmental step (expressions in words were abbreviated to letters which facilitated dealing with the expressions). In the reverse idea to phylogeny, in ontogeny most pupils have no idea why they have to use letters in algebra. This is probably the main difficulty in bridging arithmetic and algebra.

Differences in syllabuses in different countries

This depends on the profound question of what is the goal of teaching mathematics. If the goal is just seeking concrete mathematical knowledge then the differences in syllabuses are a great obstacle for the exchange of ideas in the didactics of mathematics. However if we agree that the goal is the development of meta-cognitive and cognitive abilities then the differences are of minor importance. In this case the method of teaching and questions such as motivation, class discussion, mistakes etc are the major issues to deal with.

Linkage between research and class/teacher

It is important that researchers involve the class teachers of the students/pupils with whom they are working. If the teacher feels that he/she is simply the postman for the researcher, then the interaction between experimenter and teacher has a transmissive and not constructive character. The consequence of this is that the interaction between the teacher and the class also has a transmissive character. Again the teacher wishes to be involved because the researcher is working with 'their children' and the teacher feels responsible for his/her class. In addition, the class-teacher can provide an enormous amount of information to the researcher about the previous background work which the pupils have done. On the other hand the researcher can help the teacher in his/her professional development.

Building and recognising structure

Two different research approaches were presented, one based already constructed theory (Schwank) and the other based on experimental findings from specifically designed tasks. The common aim of both these approaches was to try to explain how pieces of knowledge are connection to create a structure.

Language

Each structure is linked to a particular language(s). When a structure is changed the language usually changes. For instance, when students start to move from the spontaneous level of understanding solids to a Euclidean level they start to build new and more precise language. Vague words like side and corner are replaced by the mathematical terms edge and vertex but the term side often used with two meanings in its vague form, that of edge and also face, whereas the term corner is used in an everyday language sense. In the instructive way of teaching it is usual to start with terms without considering the concept with which those terms are linked. Frequently, this artificial ordering of terms and concepts presents an obstacle for the development of structure. We believe that for a natural development of the underlying structure, the concept should come first and the correct terminology applied as the concept is being developed.

The papers which were presented raised many fundamental issues as seen from above and the research done to create the papers goes someway to resolve some of them or at least offer suggested outcomes.

BUILDING STRUCTURE IN MATHEMATICS WITHIN TEACHING AND LEARNING PROCESSES - A STUDY ON TEACHERS' INPUT AND STUDENTS' ACHIEVEMENT

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Abstract: *Mathematical objects are multiple connected one to each other, but also with non-mathematical objects and thus build up a network with manifold linkages, characterizing the structure of mathematics. During teaching and learning processes some parts of this network are carried over into pupils' minds, changing their structure. This paper presents part of a study carried out in Germany, investigating the transformation of a network to the topic "sets of two equations of straight lines" during teaching and learning processes. Especially there are compared the network, that teachers stated to have taught in middle grade classes, and the networks, their students really learned. Overmore, there are provided findings on students' abilities in using their network knowledge for problem solving.*

Keywords: mathematical network, network knowledge, connections, structure, network categories

Introduction

In the preface of the NCTM Yearbook 1995 it is pointed out that

> One of the four cornerstones of the NCTM Curriculum and Evaluation Standards for School Mathematics asserts that connecting mathematics to other mathematics, to other subjects of the curriculum, and to the everyday world is an important goal of school mathematics. Among recent reports calling for reform in mathematics education, there is widespread consensus that mathematics … must be presented as a connected discipline rather than a set of discrete topics … (House, NCTM Yearbook 1995 – Preface.)

The notion that mathematical ideas are connected should, according to the NCTM Principles and Standards for School Mathematics 2000 (p. 64), permeate the school mathematics experience at all levels.

These demands are not new, but they are expressed to an increased extend in the last few years. Especially in Germany, the call for a reinforced representation of mathematics as a network of interconnected concepts and procedures becomes louder, not at least because of the results of the TIMS-Study (Baumert & Lehmann, 1997; Beaton et. al., 1996; Neubrand et. al., 1998) that reveal a great failure in students' problem solving abilities according to a lack of flexibility in thinking in mathematical networks. This failure was once again confirmed by the PISA–Study, where interconnections and common ideas were central elements (OECD, 1999, p. 48).

However, there hardly exists detailed information about the existing deficits. Thus there is a need to reveal the exact lacks concerning students' knowledge of mathematical networks (and with it mathematical structures and conceptual knowledge) and the reasons of the deficits. The presentation below reports on a part of a study to this concern (see also Brinkmann, 1999, 2002a, 2002b). The aim of the study was to investigate how a mathematical network as it is presented in textbooks is transformed when carried over into students' minds during teaching and learning processes, and particularly focused on the topic "sets of two equations of straight lines" in middle grade classes. This paper reports on the investigation and comparison of the network, that teachers stated to have taught, and the networks, their students really learned. Overmore there are presented some findings on students' abilities to use their network knowledge for problem solving.

Theoretical Background

The Concept of Mathematical Network

The term *network* as it is used in everyday language denotes a system consisting of some components that are manifold connected, interrelated, and so dependent from each other. Such a network can be modelled mathematically by a *graph*: the components are represented by the vertices of the graph and every connection between two components, every dependence from one component on another, is represented by an edge of the graph. If two components, a and b, are mutual dependent one of each other, the edge showing this dependence is pictorial represented by a line, or alternatively by two arrows, one connecting a with b, denoted (a, b), and one connecting b with a, denoted (b, a). If only b is dependent from a, the edge between a and b is directed and pictorial represented by only one arrow (a, b). Thus, the edge-set of a graph corresponds mathematically to a *relation* on the vertex-set, modelling the interrelations of the system components.

Mathematical knowledge has the structure of a network, as mathematical objects, i.e. for example concepts, definitions, theorems, proofs, algorithms, rules, theories, are manifold interrelated but also connected with components of the external world. Thus, a mathematical network may be represented by a graph whose vertices represent mathematical objects and nonmathematical components, and whose edges represent a relation on them, each of the edges linking the vertices of two mathematical objects or the vertex of a mathematical object and the vertex of a nonmathematical component.

Cognitive Theories

At present there exists no comprehensive theory that could describe the cognitive dimension of knowledge networks. However various theories and models are suitable for the description of particular cognitive aspects, such as aspects of manifestation of connections in the brain, the generation of networks, their storing, alteration, retrieve from memory, or the use of networks in thinking processes or problem solving processes (Brinkmann, 2002b, pp. 74-100), as for example the followings.

One of the most adequate theories is that of constructivism, with its basic principle that knowledge (in its interrelatedness) is not passively received but actively built up by the cognising subject. The social constructivism takes additionally social interaction into consideration. The constructivism provides a model suitable to explain the generation and alteration of individual cognitive knowledge networks.

Theories of situated cognition point out the situated nature of learning, remembering and understanding. According to these theories, knowledge is always connected to the context of its acquisition. This context is mostly of nonmathematical nature, it is stored in mind together with the respective knowledge and influences its activation, its use and transfer.

Network Categories

Mathematical objects may be linked in very different ways one to each other or to the external world. The different sorts of linkages define different relations on sets of mathematical objects respectively between mathematical objects and nonmathematical components, and thus different *network categories*. Main mathematical network categories with relevance for mathematics education in school are defined by Brinkmann (2001c, 2002b). The study part presented here restricts on some relations on sets of mathematical objects, i.e. *relations according to subject systematics* and a special relation according to the application of mathematical objects, the *model relation*.

Essential relations according to subject systematics and the linking aspects defining them are:

- different interpretations of the inclusion relation (*part-whole link, subset-superset link, subconcept-superconcept link, case distinction link, classification link, characteristic/feature link* (i.e. link between a characteristic/feature of a mathematical object and this object)),
- relation of deduction (*deduction link*, i.e. link between a mathematical object and another deduced from it),
- relation of belonging (*belonging link*, i.e. for example link between a theorem and a proof of this theorem, link between a problem and its solution).

The model relation on a set of mathematical objects is given by *model links*, i.e. links between two different mathematical representations (for example a geometric representation and an algebraic representation) of the same mathematical object, in order to get solutions for a mathematical problem using representational change.

Graphical Representation of Mathematical Knowledge

When we want to analyse mathematical knowledge in its interrelatedness it is appropriate to represent this knowledge in graphs. Methods to transform texts (out of textbooks, or transcripts of interviews) respectively interrelated contributions given in an interview, a discussion or a conversation (e.g. in mathematics lessons) into a graphical structure are described in Brinkmann 2001b, 2001c, 2002. In order to map out an individual's knowledge of mathematical networks graphically, concept mapping (see e.g. Novak & Govin, 1984; Novak, 1990, 1996) turns out to be a suitable means. It uses special two-dimensional graphs - named concept maps - showing the concepts related to a given topic together with their interrelations.

Methodology

The participants of the study were 3 experienced teachers, regarded by their colleagues as being very proficient, of different schools in Germany (2 gymnasiums and 1 comprehensive school) and altogether 137 of their students (6 courses: 5 courses of the 2 gymnasiums and 1 course with higher achievers of the comprehensive school), out of grades 8, 9, 10 and 11. The 8^{th} and the 9^{th} graders had just been instructed in the focused subject, the 10^{th} graders had learned the subject one respectively two years ago, and the 11^{th} graders three years ago. Every course dealt several weeks with the topic focused in the study. At the moment, when the teachers were asked to participate together with their classes at the study, the respective teaching unit was already finished. Thus no special educational style with regard to the study was used, but only *usual traditional education*. Characteristically, new knowledge is either presented by the teacher or developed in a question-response sequence, in which the teacher poses mostly narrow questions that require short answers.

Teacher interviews

The teachers' statements in respect to the network around "sets of two equations of straight lines" they implemented in classroom were investigated by interviews, following the questionnaire given below (table 1). According to the network modelling by a graph, vertices (i.e. teachers' statements about concepts they presented in classroom) and edges (i.e. how these concepts have been linked to each other) had to be revealed. The conception of the questionnaire took into account that classroom teaching is mainly based on textbook contents: correspondences and differences were asked. The questionnaire follows the principle of putting first broad questions that are subsequently narrowed down. Dependent on the answers given by a teacher during the interview some supplementary questions revealing further details should be formulated. The fifth question serves getting information about special, extraordinary facts that may have taken influence quite out of the ordinary on the students' achievements.

Table 1: Questionnaire

1. Which contents were taught? Did the choice of the contents follow the textbook and if so, to which extend? Did you teach also contents that aren't represented in the textbook?
2. How did you integrate the contents?
3. Which solving algorithms did you introduce to the students? Did they learn also graphic solving methods?
   Have algebraic representations of pairs of simultaneous linear equations and solutions of pairs of simultaneous linear equations been interpreted geometrically? ...

4. Which links between mathematical contents were shown? Did you deal exactly with the connections presented in the textbook; where did classroom presentation differ from that in the textbook, what was left out, what was replaced, what was added? …
5. Did the students show any special affective attitude? ...

Student tests

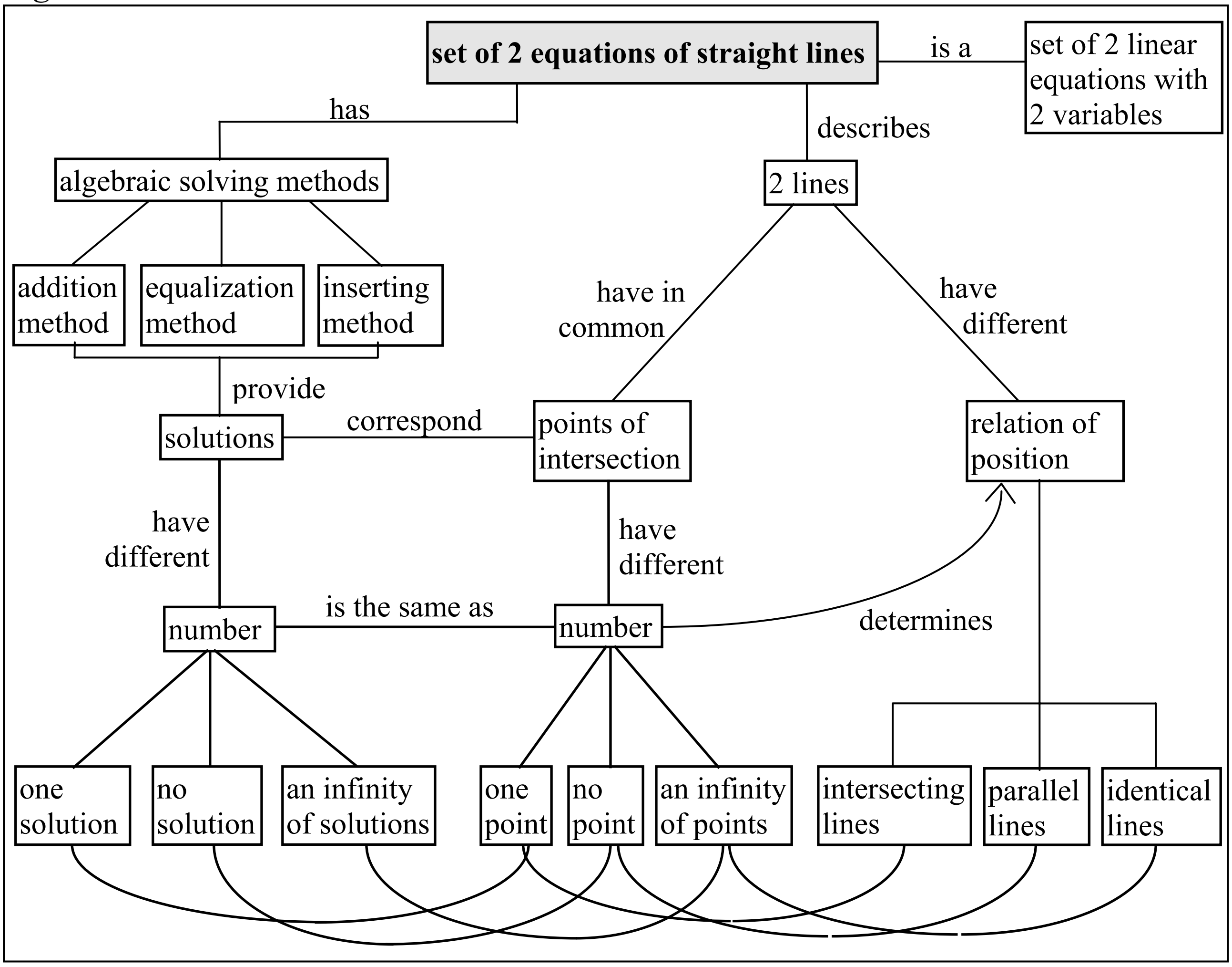
In order to map out the networks learned by students, concept maps drawn by them would have been a suitable means. The use of concept mapping requires the students to be familiar with this method, this condition was not given. Thus, the author developed students' tests demanding activities closely related to those of concept mapping, without asking the students to draw a concept map by themselves:

- First the students had to write down all the concepts that they connect in some way with "pairs of simultaneous linear equations". This brain storming indicates predominant links in students' minds.
- As it is possible that students are aware of further links to concepts that they didn't remember in the brain storming process, they received a list of concepts in the second test step, with the task to mark those ones that are linked in some way with the topic-concept. The selection of concepts to be listed, had to take into account what students could know according to the presentation of the focused topic in the classroom lessons, thus research results of the teacher interviews had to be considered here. Concepts which are in no way connected with the topic-concept were chosen that way, that they couldn't be simply guessed. If the students knew further concepts linked to the topic-concept, but not listed, they were asked to complete the list with them.
- By the third test item the students had to build conceptual classes with concepts of the list from the second item and to give for every class a generic term. This item reveals more detailed information about students' knowledge of existing linkages and gives an insight in the way how students' conceptual knowledge is structured.
- To get more details about the knowledge of some special links, an incomplete concept map was given to the students in the fourth test item, with the task to fill in missing concepts and links. The most part of this concept map is shown, in its completed form, in figure 1[1]. It represents major concepts implemented in classroom together with the relations between them.[2] The missing concepts and links were some at the bottom of the map. Thus, this item examines especially the knowledge of model links between mathematical concepts but also of links according to subject systematics between the concepts in the last concept-row and those in the row above.

[1] The concept map used included in addition different algebraic structures of a set of 2 equations in straight form, and their relations to concepts shown in figure 1.

[2] The concept map shown in figure 1 was constructed by the author on the basis of the outcomes of the teacher interviews; see figure 2, first graph.

Figure 1



For the investigation of students' abilities to use their network knowledge, some problems were posed in a fifth test item (table 2). These problems were of a new type for the students so that they could not be solved just by already practised algorithms, rather flexible thinking in networks was demanded.

Table 2: Problems

1. The equation $3x + 2y = 7$ describes a straight line g. Write down an equation for another straight line h, that
   a) is parallel to g,
   b) intersects g in the point P(1; 2),
   c) has more than one intersection point with g.
2. The straight lines g_1 and g_2 are parallel. The straight line h intersects g_1 in the point P(2; 1) and g_2 in the point Q(0; 3). Write down equations that may describe the three straight lines.

Networks

Out of the data obtained by the teacher interviews and the first four student test items the corresponding networks were represented graphically, and compared (for more details on the methodology used here see Brinkmann, 2002b). In the graph based on the outcomes of the student tests only those links were drawn, that were achieved by at least 50% of the students.

Research Results

Results of the teacher interviews – brief outline

All three interviewed teachers answered that their classroom implementation to the topic "sets of two equations of straight lines" followed nearly exactly the textbook, referring the represented concepts as well as their connections. In particular, there were discussed the different relations of position of 2 straight lines (case distinction link), and correspondingly the different number of intersection points (case distinction link, model link), respectively the different number of solutions of a set of two equations of straight lines (case distinction link, model link). Further there were introduced several solving algorithms for sets of 2 linear equations with 2 variables (subconcept-superconcept link). The teachers expressed that most of the students understood and learned the presented connections between algebra and geometry, but that the solving algorithms are the best and lasting learned. After some time, they said, most of the students essentially remember the solving algorithms, connections between algebra and geometry get first lost.

Results of the student tests

The first test item reveals that the students connect in a sensible way several geometric and algebraic contents with the topic, e.g. "graphs in a coordinate system"/ "lines" (54%), "addition method" (19%), "equation of the straight line" (16%), "linear function" (15%).

The results of the second item show that most of the concepts that are linked with the topic-concept were also marked by the majority of the students and concepts that have no connection to the topic-concept were respectively marked only by a minority of the students. In particular, it turns out that most of the students know the names of suitable solving algorithms. Further it is obvious that students know about some connections between geometry and algebra, as most of them marked correctly algebraic concepts as well as geometric concepts.

By working on the third test item the students showed various classification aspects for concepts linked to the focused topic. The generic terms named most frequently were: algorithms (51%), lines (45%), solutions (32%). Most of the concepts connected with the single generic terms were correctly subsumed. Right solving algorithms were named by about 80% of those students who gave "algorithms" as generic term. Under the superordinate concept „lines" there were subsumed a lot of concepts that might also be classified in a more differentiated way. Most of the conceptual classes given by the students show linkages according to a subconcept-superconcept relation, i.e. simple links according to subject systematics. Model links between geometry and algebra were pointed out only by one student under the generic term „solutions".

The fourth test item reveals a great uncertainty in the knowledge about model links between mathematical concepts. The students were more successful in finding the right missing concepts than the missing links. Only 5% of the students managed to complete correctly all the missing elements of the concept map. The concept completions show that about half of the students know about the different possible number of solutions of sets of 2 equations of straight lines and also about the different possible number of intersection points of lines (case distinction link). The knowledge about the different relations of position of 2 lines appears only by 30% of the students.

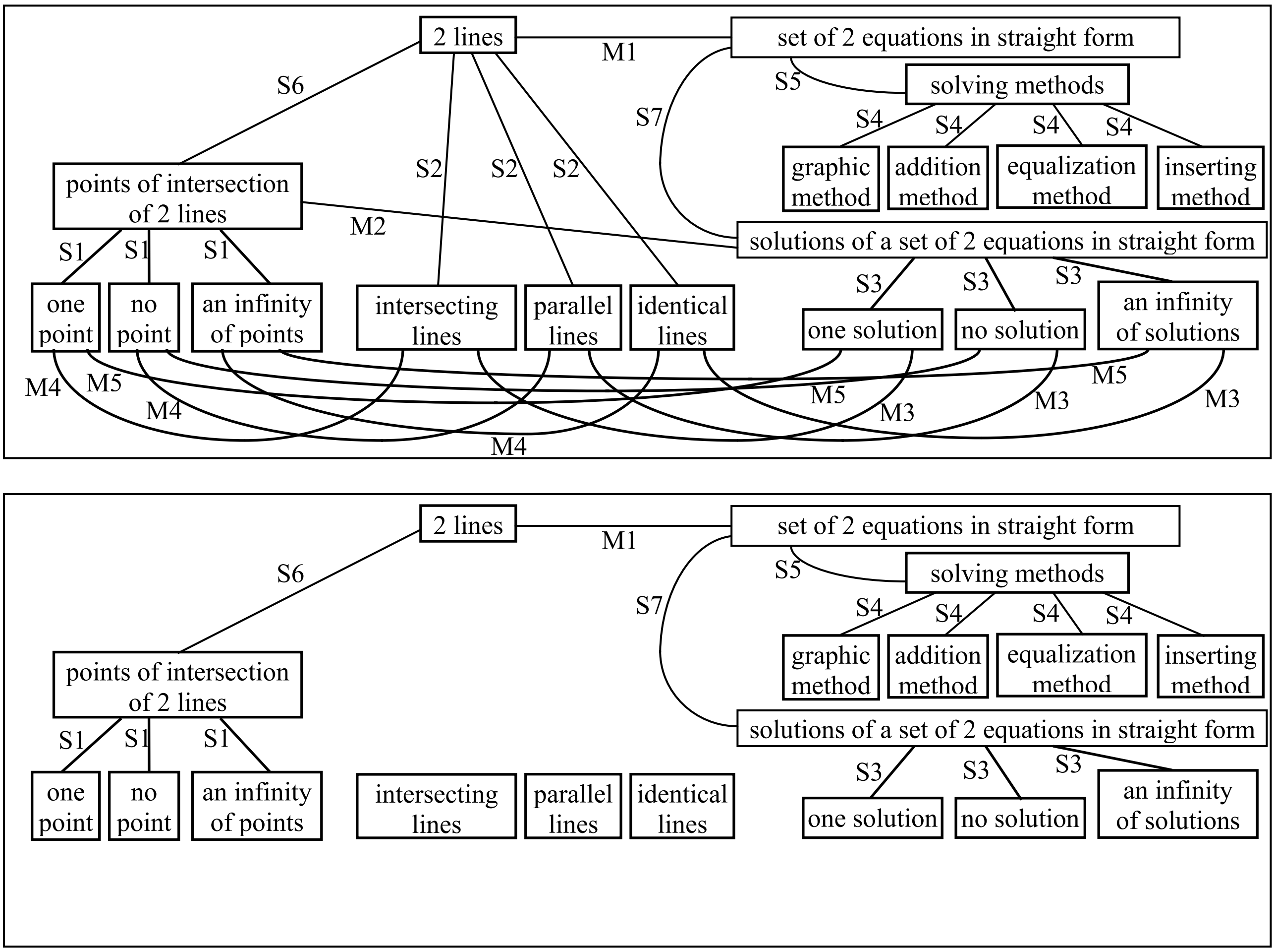
The problems of the fifth test item were solved only by a minority of the students (problem 1a by 22%, 1b by 9%, 1c by 27%; problem 2 by 5%). Even those students that appeared to have the necessary conceptual knowledge, partly failed in solving the posed problems, especially problem 2.

There were no significant differences between the results obtained from students who had just dealt in classroom with the focused subject (8th and 9th graders) and from those who learned this subject longer time ago (10th and 11th graders).

Networks

The following graphs (figure 2) represent outcomes of the investigations. The first graph shows the network teachers stated to have implemented in classroom, the second graph visualizes, according to the outcomes of the first four test items, which of the network links are achieved by the majority of the students. The different links according to subject systematics are marked with S, model links with M.

Figure 2



A comparison of the two networks brings out that the mainly learned connections by students are part of the links according to subject systematics, model links are hardly known. The manifold interrelations of the concepts, that students associated in the second test item with the topic-concept, are for a great deal lost. The students know about isolated concepts around the topic, but these are not well-structured.

Discussion

The study reveals the incompleteness of the transfer of implemented networks in students' minds, in usual traditional education in Germany. Moreover, the missing relations in the achieved networks are pointed out and characterized; thus we have more differentiated and exact results than provided by the great studies TIMSS and PISA.

Consequences in respect to the teaching of connections have to be discussed. According to the theory of constructivism, the students have to be given tasks so that they can discover connections by themselves. In particular, these tasks should develop linkage and understanding between the algebraic and geometrical representation of simultaneous linear equations, as the study indicates lacks here.

Furthermore we should use more adequate methods for representation of mathematical networks in classrooms. Concept maps and mind maps seem to be efficient tools for this purpose (see e.g. Brinkmann 2001a, 2002c, 2003a, 2003b), not at last because of their graphical structure with network character.

The study presented in this paper shows also, that even those students having the necessary conceptual knowledge are not necessarily able to use it successfully in problem solving processes. One reason might be the presentation of the focussed topic in the textbooks followed by the teachers. Although there is given a description and explanation of the interrelatedness of concepts connected with simultaneous linear equations, there is a great lack of problems demanding this network knowledge for solving them. Most of the problems given in the textbooks can be solved by using algorithms, and thus these are the problems mainly treated in lessons, homework and exams. An increased number of purposefully constructed problems requiring flexible thinking in networks should be offered and probed in classrooms.

In order to evaluate the efficiency of respective changes in the educational stile, in the methods for representation of mathematical networks, or in textbook presentations, further studies have to be carried out, including also direct observations of teaching processes.

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THE METAPHOR "CONTRACTS TO DEAL WITH CONCEPTS" AS A STRUCTURING TOOL IN ALGEBRA

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Abstract: *Many students' mathematical knowledge is fragmented and they do not perceive the links between these fragments. A Cognitive Mathematics Education approach, i.e. an orientation on the thinking processes, gives a chance for change. We report about the main ideas and the outcome of a curriculum project, in which the construction of a cognitive mathematical operating system in the pupils' heads is put into the centre of our conceptual work. Metaphors play an import role in establishing the system. The aim is that in the beginning of any maths lesson this operating system boots in pupils minds and organises the connection of the mathematical knowledge. This hinders fragmentation. We present examples of pupils' work and analyse how the operating system controls the organising of mathematical knowledge.*

Key-words: Cognitive Mathematics Education, metaphors, algebra, metacognition, classroom culture.

1 Introduction: Cognitive Mathematics Education

Many students' mathematical knowledge is fragmented and they do not perceive the links between these fragments, since they are stored in the student's mind using memory only. In order to remedy this bad state of affairs, it needs a new orientation of mathematics teaching and learning. We have moved the main focus away from a subject-orientation to a cognitive theoretical basis. It means that the cognitive mechanisms when constructing mathematical knowledge in the pupils' heads and the process of knowledge organisation and knowledge use form the centre of the teacher's attention. Our approach "Cognitive Mathematics Education" (CME) combines a curriculum branch (Osnabrueck Curriculum) with a change in classroom culture, based both on research in cognitive science.

- The Osnabrueck Curriculum (OC) puts the construction of a cognitive mathematical operating system in the pupils' heads in the centre of the conceptual work instead of teaching fragmented mathematical expertise which remains isolated. Its most important elements are
  - the function frame and
  - the contract frame

  with suitable, attached procedures. It includes the development of suitable learning environments. The aim is that, in the beginning of any maths lesson, this operating system boots in pupils minds and organises the connection of the mathematical knowledge. We have followed this guideline when conceptioning OC for grades 7 to 10 (age 12 to 15) at schools of the type "Gymnasium"[1]. (Cohors-Fresenborg, 2001; for an overall view of the ideas in English see Cohors-Fresenborg, 1993b; Cohors-Fresenborg, Schwippert and Klieme, 2000).

- The classroom culture is determined by
  - an orientation on the pupils' conceptions, discussing the relationship between external and internal representations,
  - an acknowledgement of a discursive teaching-learning style,
  - a metacognitive behaviour of teachers and pupils (Cohors-Fresenborg & Kaune, 2001),

  which all three are supported by new formats of exercises, tests and textbooks.
- The basic research is concentrated on the concept formation processes, especially
  - individual differences in cognitive structures, predicative versus functional thinking (Schwank, 1993),
  - the role of mental models and specific learning environments.

The question as to whether the considerable shift of emphasis necessary for the achievement of these changes – the contents of the present curriculum have to be taught in approx. 75% of the teaching time – is at the expense of the performance in achieving the learning targets of the regular curriculum or whether the investment in the construction of an operating system already pays off after a short while.

In order to examine this question, a comparative study with TIMSS-instruments had been undertaken (Cohors-Fresenborg, Schwippert and Klieme, 2000). In summary, it can be said that the construction of the above-mentioned cognitive mathematical operating system makes it possible to make up the necessary teaching time as early as at the end of grade 8 and to achieve – at the individual level – significantly better competitiveness when referenced against the TIMSS-scale. One conclusion is that the "amount of teaching contents" is not the main hindrance to the improvement of mathematics teaching, but that the quality of teaching has to be improved.

This paper gives a first insight into how one of the two parts of the cognitive operating system, the "contract metaphor", is developed in the pupils' heads with the aim that future knowledge is not gained in fragments, but is knotted into an already existing semantic network. The metaphor "operating system" is meant to make clear that the organisation of access is an important task when constructing a knowledge network. With the help of case studies, we further analyse what role the operating system plays whenever pupils work on algebra-problems and proofs. This analyse shall also theoretically explain to which cause we put down the discovered increase in performance, especially with average and weak pupils, in the above-mentioned empirical study: the knowledge is not fragmentary, the access to knowledge is organised.

[1] The *Osnabrueck Curriculum* is a result of two long-lasting curriculum projects in the German Federal State of Lower Saxony for the improvement in mathematics teaching in schools of the type "Gymnasium". In Germany the school system at secondary level is divided into three levels. A "Gymnasium" is the school of the highest level. In Germany about one third of each age group attend a "Gymnasium".

2 The Metaphor "Contracts" as part of a Cognitive Mathematical Operating System

The description of the mathematical knowledge of pupils via the concepts "frame" and "procedure", introduced to the cognitively-oriented mathematics education by Davis and McKnight (1979), gives cause for searching fundamental frames and procedures. These have to be developed as parts of a mathematical cognitive operating system in the pupils' heads in the first lessons at grammar school and their combination has to be consciously implemented for the pupils.

The "function frame" and the "contract frame" are the most important parts of this mathematical cognitive operating system. Both use the frame "formal representation of intuitive knowledge". This requires a fundamentally new judgement of the importance of language and formalisation in mathematics teaching from the teachers, and moreover justifies itself by the increase in value of understanding and communication. For the introduction of the function frame in grade 7 see Cohors-Fresenborg (1993a, b), Kaune (1995) and Sjuts (1999a, pp. 77-91). It is important to notice that the "function frame" is not simply a knot of subject knowledge in a knowledge net, but part of the cognitive operating system which organises, controls and supports the use of knowledge.

Mathematics teaching, especially at "Gymnasium" level, is meant to give pleasure in the theory of, and in the capability to use, mathematics. Here a deep understanding of the formation of abstract mathematical concepts is indispensable. Our starting-point offers easy access to this by the question what is actually meant by the intuitively existing mathematical concepts: The cognitive theoretical procedure only has to concentrate on mathematics itself. Thus it is clear that questions of meta-mathematics, the formation of concepts by and in axiomatic systems and the nature of precise and explicit definitions and proofs become an integral part of the lessons. The axiomatic method is a means in order to generally organise a (mathematical) stock of knowledge, to ensure it in itself and to develop it. It has been achieved to give even pupils of grade 8 a long-term workable framework idea by establishing a suitable microworld "Sentences from the Desert"[2]. Following that the microworld „Contracts for the Dealing with Numbers" (textbook and teacher's manual: Cohors-Fresenborg et al., 1998) offers a frame which provides a unified basis for insight into the formation of concepts in customary school mathematics in fields such as extension of the number domain, term rewriting, equations and the method of proving. Probability calculus also fits this frame: It is a contract to talk precisely about the uncertain (textbook: Cohors-Fresenborg et al., 1994).

Just like new software is installed with the help of an operating system in the world of computers, the cognitive operating system conceived by us enables the pupils to integrate new mathematical theories and views, such as extension of number domain

[2] The text book and a detailed teacher's manual (Cohors-Fresenborg et al., 2003) unfortunately only exist in German, but a short English description of the didactic concept can be found in Cohors-Fresenborg (1987, pp. 268-271).

and probability calculus, into their existing knowledge network already at the time of gaining this knowledge.

3 The Benefit of the Metaphor "Contract" in School Algebra

3.1 The Introduction of the Metaphor "Contract" in Lessons

"Contract for Dealing with Concepts" is a metaphor (in the sense of Lakoff, 1980) used in the Osnabrueck Curriculum for the axiomatic dealing with concepts. In our teaching series "Sentences from the Desert" we do not introduce axiomatic systems by abstraction from many examples, contrary to the common procedure in German-speaking mathematics education of the sixties and seventies. We do, however, use a framework story, "Contract for the Construction of Motorways", (Cohors-Fresenborg et al., 2003, p. 33), in which a sheikh concludes a contract with a building-company for the construction of a motorway network in his emirate. The discussion during the lesson is about what minimum has to be built according to this contract (mathematically speaking, which characteristics each model of the axiomatic system must have).

We also want to impart to the pupils that the process of precisely stating concepts by defining has to end sometime with an amount of concepts, for the precise stating of which other tools are needed: The extent of meaning of those concepts looked at in relative terms to other fundamental concepts is regarded as evident in the argumentation context. A reflection of syntax and semantics clarifies the connection between a (mathematical) object, its name and a sign for its name. It provides a workable imagination of what a writing figure means, which presupposes the existence of an object (Sjuts, 1999a, p. 123).

By calling up the metaphor "Contract" we use the pupils' intuitively existing knowledge about the problems of using the concepts in juridical contracts. Through this we also create access to the understanding of the difference between implicit and explicit definitions. Proofs in a contract also serve to make sure that the implicit definition of a concept network by the contract (axiomatic system) meets the intention.

In the following we first briefly describe how the concept of "whole numbers" is introduced in lessons in grade 8 following the Osnabrueck Curriculum. Then we give two examples, one concerning linear equations (in grade 8), one concerning term-rewriting procedures with roots (in grade 9). Each example first presents a problem from a written class test followed by some pupils' solutions (for partial problems), which are interpreted and analysed within our conceptual framework.

3.2 Introduction of a Contract to Deal with "Debit and Credit"

The availability of the metaphor "Contract for Dealing with Concepts" allows a different procedure for dealing with the extension of the number domain of natural numbers to whole numbers: There is a contract to be concluded with a bank in order to book “Debits and Credits”, which guarantees that the bank books exactly according to the customer’s ideas in every case. This means that existing intuitive knowledge about the booking practice of financial institutes when running an account (i.e. calculating with positive and negative numbers) has to be precisely stated and the procedures of paying in and out have to be formally described. The result is a contract to deal with „Debit and Credit”, mathematically speaking, an axiomatic system for commutative groups. It is also worked out to what extend the paying out process can be taken as a reversal function of the paying in process. The nominal definition of subtraction resulting there from is cited as “*S”* by the pupils in term rewriting (as well in the transcription below).

As the pupils have already gained considerable experience with formal representations from the teaching series “Sentences from the Desert”, this does, of course, also include for the pupils that the final version of group axioms have been put down in writing in a formalized representation in predicative logic. The pupils’ solutions 3.3.2 show that the pupils have not gained fragmented expertise, but a competence which they will use themselves whenever they formulate new facts and have to discuss them.

3.3 Application in the Field of “Linear Equations”

The worked out axioms for fields, understood as a contract for calculating, constitute a tool to justify term rewritings. In order to solve linear equations the contract has been enlarged by two paragraphs: $\S\, R^{+}$ justifies the addition of a term on both sides of an equation, $\S\, R^{\cdot}$ justifies the multiplication.

3.3.1 A Problem from a Written Class Test

Pupils of grade 8, who had been taught according to the Osnabrueck Curriculum since grade 7, were confronted with the following problem in a written class test:

<u>Problem</u>: *Paula exercises the solving of linear equations with the following example:*

$$1{,}25y - 0{,}5 \cdot (1 + 0{,}5y) + 2{,}5 = -4{,}25 - 0{,}5 \cdot (y - 0{,}5) + 0{,}5y$$

After a couple of equivalent rewritings she has simplified the equation to: $y + 2 = -4$ *With the following comment she gives up:*
”I must have made a mistake. We have never had such equations. You will only get further if you are allowed to use a paragraph R^{-}*.”*
a) Check if Paula has made a mistake.
b) Formulate a paragraph R^{-} *according to what Paula has in mind.*
c) What would you reply to her?

This problem is typical for the Osnabrueck Curriculum and the problem culture practiced therein (see Kaune, 2001a): The pupils are prompted to reflect their own ideas and those of their class-mates and then to put their views down in writing. Such metacognitive activities (see Sjuts, 1999b; Cohors-Fresenborg and Kaune, 2001) also prepare meta-mathematical understanding.

The partial problem ‚a' asks for a term rewriting and a justification by paragraphs. Partial problem ‚b' asks to analyse at first the concrete situation in a general light and then to generally formulate the knowledge gained as a paragraph. This means, the generating of universally usable new knowledge is demanded. Mathematically this means the invention of a new theorem. Partial problem 'c' is about the discussion of the theorem's place value in the previous contract.

3.3.2 Pupils' Solutions

Partial Problem b

Tanja: R^{-} $\bigwedge_{a}\bigwedge_{b}\bigwedge_{c}\bigwedge_{d}$ $a+b=c \Leftrightarrow a+b-d=c-d$

Henrik: R^{-} $\bigwedge_{a}\bigwedge_{b}\bigwedge_{c}$ $a=b \Leftrightarrow a-c=b-c$

Frank: R^{+} $\bigwedge_{a}\bigwedge_{b}\bigwedge_{c}$ $a=b \Leftrightarrow a+(-c)=b+(-c)$

Partial Problem c

Tanja

"How do you know that you have made a mistake? Instead of $y+2=-4$ you can also say $1\cdot y+2=-4$ and we have always had such equations. Furthermore we have never needed a § R$^-$ and will not need it, as this problem can be solved by saying e.g. $+(-a)$".

$$
\begin{aligned}
& y+2=-4 & \\
\Leftrightarrow\ & y+2+(-2)=-4+(-2) & \text{§ R}^{+} \\
\Leftrightarrow\ & y+0=-4+(-2) & \text{§ I}^{+} \\
\Leftrightarrow\ & y+0=-6 & * \\
\Leftrightarrow\ & 0+y=-6 & \text{§ K}^{+} \\
\Leftrightarrow\ & y=-6 & \text{§ N}^{+}
\end{aligned}
$$

Henrik

We have already got this § indirectly, because with the help of *S* we can change $a-b$ to $a+(-b)$ and also vice versa. This means we can calculate with R$^+$, e.g. $a+(-c)=b+(-c)$, and then we can rewrite this as $a-c=b-c$ with the help of *S*.

Frank

We have already got a sort of paragraph R$^-$. It belongs to paragraph R$^+$. It is used according to the principle of paying debts.

3.3.3 Interpretation

All three pupils' solutions have to be considered correct, but there are quality differences regarding their depths of understanding.

Tanja's formalization is very much oriented to the example's term structure. She starts from terms in the form of a sum, she solves, however, the problem of subtraction of an arbitrary term. In her argumentation she furthermore shows that the problem can be solved without the use of a new theorem. We consider this an initial kind of proof for her theorem.

Henrik states a general solution and gives reasons, why this new theorem from the previous contracts is provable.

Frank states the general solution under a name and in a description which provides the reader immediately with the idea for a proof. Apart from this, his argumentation also makes clear what use the metaphor „credits and debits" has for his understanding.

3.4 Application in the Field of "Term-Rewriting with Roots"

The Osnabrueck Curriculum deals with all term rewriting rules in the way documented in the last paragraph. The distinction between syntax and semantics as a subject of the Osnabrueck Curriculum on one hand allows to prove facts in axiom systems and then to use them for different interpretations. On the other hand it allows the discussion of the problem with the pupils to what extend certain „terms" make sense or only pretend to be meaningful and therefore only look as if they were terms.

3.4.1 A Problem from a Written Class Test

Pupils in grade 9, who had been taught according to the Osnabrueck Curriculum since grade 7, were confronted with the following problem:

<u>Problem</u>: *Four pupils are discussing the solution of the following problem:*

Simplify the following term as far as possible: $\sqrt{\sqrt{18}-4{,}5}\cdot\sqrt{\sqrt{18}+4{,}5}$.

Silke: "First I thought the solution would not work. But now I know it has to be $-\sqrt{2{,}25}$ *and that is* $-1{,}5$*."*

Eva: "This is not possible, as the solution is syntactically wrong."

Michaela: "It is not only the solution that is wrong, the first line must be wrong as well."

Ariane: "Every line is wrong. I think we should not even have started calculating ... "

a) Which of Silke's comments do you agree to?
b) Why does Eva think the solution is syntactically wrong?
c) Do you agree with Michaela?
d) Please assess Ariane's comment.

This problem as well is typical for the Osnabrueck Curriculum and the different problem culture practiced therein. The construction principles and the intended effects of such problems are shown in Kaune (2001b, p. 44).

In partial problem ‚a' a typical pupil is addressed to. Part ‚b' also helps to direct the pupils' focus to the problem of the meaninglessness of certain formal writing figures.. The partial problems ‚c' and ‚d' are meant to teach the pupils to reflect more clearly where the problem lies.

3.4.2 Pupils' Solutions

Partial Problem a

Markus

$\sqrt{\sqrt{18}-4{,}5}\cdot\sqrt{\sqrt{18}+4{,}5}=\sqrt{\left(\sqrt{18}-4{,}5\right)\cdot\left(\sqrt{18}+4{,}5\right)}=\sqrt{18-4{,}5^2}=\sqrt{-0{,}5}$

The problem is syntactically wrong as there is a name given for $\sqrt{-0{,}5}$ in every line, but $\sqrt{-0{,}5}$ does not exist as a name for a number.

I can only agree to Silke's comment, i.e. $-\sqrt{2{,}25}=-1{,}5$. This means the solution of the problem is not $-\sqrt{2{,}25}$, and she should have explained more clearly that this problem cannot be solved.

Jutta

I agree with Silke when she says the problem "does not work", as there is no $\sqrt{-2{,}25}$, because it does not matter if you multiply a positive number by a positive number or a negative one by a negative number. You also get a positive number, never a negative one. Furthermore Silke probably thought that $\sqrt{-2{,}25}=-\sqrt{2{,}25}$, which is not the case.

Marion

Silke's first presumption is correct, as $\sqrt{-2{,}25}$ is not a name of a number. There is no number which - multiplied by itself - is $-2{,}25$.

Partial Problem c

Markus

Yes, I agree with Michaela. As the solution is syntactically wrong, there is also a name for the solution in the previous lines, which does, however, not exist as a name of a number.

Jutta

Michaela is right, because if you have calculated correctly, the last line is the same as the first, and if the last line is syntactically wrong, then, consequently, the first line is wrong as well.

Marion

The first line and the one in between must be wrong, too, as we have always drawn conclusions starting from the first line, and there are also the equals signs, and every line is covered by a paragraph.

Partial Problem d

Markus

Ariane is right in saying that every line is wrong. You were, however, allowed to get on calculating, as re-writings have been made in every line which can be justified by rules, laws and formulas. Moreover, it was not recognizable in the beginning that the problem is syntactically wrong.

Jutta

Ariane is also right because when you have calculated correctly, there is the same in every line, only under a different name. And then, if one line is syntactically wrong, all the others are also wrong. She is also right in saying we should not have started calculating, because if the problem is syntactically wrong, all the rest will also become syntactically wrong.

It is, however, understandable that we started calculating, as it could not be seen at first glance that the problem is syntactically wrong.

Marion

It is allowed to start calculating, because it cannot be recognized straight away that the term is syntactically wrong. You only notice that when you have come to the end of the calculation.

3.4.3 Interpretation

Presumably Markus makes two mistakes at the end of his calculation according to part ‚a'. The wrong result, however, allows him to continue his work on the problem according to the formulation of the problem. With his formulation: As the solution is syntactically wrong" he takes up his knowledge from his maths lessons (see Cohors-Fresenborg et al., 2003, p. 51): As there is no object with the characteristics mentioned, access with the help of an indication operator – as used in the definition of the root function – is not allowed, therefore he uses the formulation „syntactically wrong".

The problem, to what extend a meaning and therefore a name (for a number) is given by the supposed root term, is more clearly mentioned by Jutta.

Marion gives reasons that the syntax mistake of the last line must also have been in the first one. The view of calculating by using paragraphs of a contracts becomes very clear in her argumentation.

While Markus and Marion thought that they were allowed „to start calculating", Jutta realizes that actually they should not have started calculating, but that this knowledge could only have been gained after calculating. Jutta has got a remarkable understanding of the argumentation pattern of indirect proofs.

4 Summary

We have shown how to counteract the splitting into fragments of the pupils' mathematical knowledge: by means of the construction of a cognitive mathematical operating system in the pupils' heads in grades 7 and 8. Its two components "function frame" and "contract frame" bring together essential contents and methods of school mathematics. We have explained how calculating, term rewriting, defining and proving are linked with each other. Pupils as well have this insight as can be seen in the following dialogue in grade 10:

Teacher: "What do you have to know when you want to prove something?"

Clemens: "Yes, I think the most important thing is to know the contract, so that you know right from the beginning what you are allowed to do and what not."

Teacher: "Do you think proving is more difficult than calculating?"

John: "If there were a term, you wouldn't be able to, couldn't do it any different than that. [*He means term rewritings do not work any different than proving does*]. You would have to get it from somewhere, your knowledge from the contracts. And that would not be any different to working with variables. [*He means the representation of term rewriting rules with the help of variables*]."

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FROM EXPERIENCE, THROUGH GENERIC MODELS TO ABSTRACT KNOWLEDGE

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Abstract:*The theory of generic models (or TGM) as a concept development theory is described and compared to the reification theory. In TGM, the knowing process is decomposed into generalization and abstraction levels with the generic model as its pivot term. TGM is applied as an educational and/or research tool to: 1. characterize a teaching style and 2. create a teaching scenario (of a particular mathematical topic). Within these two issues, we consider a students's 3. discovery of relation (rule, formula, procedure), 4. concept development process, 5. non-standard solving process, 6. failure.*

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Key words: generalization and abstraction level, generic model(s), isolated model(s), knowledge with understanding, mechanical knowledge, structurisation.

INTRODUCTION

A transmissive teaching method based on transmitting ready-made knowledge from a teacher to students causes that a student's knowledge often suffers from the lack of understanding. Such mechanical knowledge is kept in memory without connections to the student's life experience or other areas of his/her knowledge structure. Such isolated knowledge can only rarely be used (except in standard situations such as drill) and is quickly forgotten. The theory described and used in this paper was elaborated in the 1970th by the first author under the guidance of his father V. Hejný. It was first published in English in (Hejný, 1988) and later elaborated further in several papers, e.g. (Hejný, 2003). This paper presents our current understanding of the pivot term of TGM, i.e. a generic model, which gave the theory its name.

THEORETICAL FRAMEWORK

Theory of Generic Models

The structure of TGM is depicted in the following scheme:

experiences → generalization → [generic model] → abstraction → abstract knowledge (i)

It consists of two consecutive levels. In the first, Generalization level, i.e.

experiences → generalization → generic model, (i')

students' experiences group and change into one image – the generic model. In the second, Abstraction level, i.e.

generic model → abstraction → abstract knowledge, (i")

the generic model loses its embedment in object thinking and changes into abstract knowledge whose structure is richer than that of the generic model.

Generalization level (i'). Each concrete experience is stored in memory as a model of future knowledge. At first, each of the models is isolated but later on, some of them start to *refer to each other* (see the example below). When the web of these bilateral references reaches a certain density, the set of models, so far isolated, is arranged and *changed into a group*. The references are often of two types: congruent and contradictory. The congruent ones, which create the group, are changed into a new mental object (concept, relationship, rule, scheme, etc.). The contradictory ones help to define the boundary of the new mental object. The mental object is a generic model which plays a role of a representative of the whole group. In some cases, the final generic model arises from particular generic model(s) via amalgamation.

For example, a child observes that two sweets and one sweet give three sweets, two dolls and one doll give three dolls, etc. At the beginning, there is no linkage among these experiences. However later on, a child starts to see that there is something common in all these situations and finally he/she recognizes the fact that two fingers and one finger give three fingers is the same as cases with sweets, dolls, etc. Fingers become a generic model, or even a generic tool not just for this particular knowledge, but also for many arithmetical situations. The described process is generalization terminating in the knowledge which we will refer to as the generic model of the future abstract knowledge 2 + 1 = 3. For more detail, see (Hejný, 2003) where the term 'universal model' is used instead of the term used here, i.e. 'generic model'.

Abstraction level (i"). On one hand, the generic model covers a wide area of object experiences, on the other hand, the generic model remains an object representation and does not allow for a higher level of structurization of acquired knowledge. Therefore, the next step of knowledge development must be abstraction, i.e. disconnection from an object characteristic of a generic model. This shift is accompanied by a change of language and an object representation is enriched by a symbolic representation.

Generic model. We have seen that the generic model is a pivot term in scheme (i). Within (i'), it is the final stage of generalization and at the same time within (i"), it is the source of abstract knowledge. As the former, the generic model is a representation of all corresponding isolated models. It may replace any of them if needed. If the generic model covers just a part of all isolated models it is not sufficient for solving the whole variety of problems (see Illustrations 3, 4 and 5). As the latter, the generic model plays the same role as the isolated model with respect to generalization. Many of our experiments proved that knowledge is mechanical if and only if there is no generic model linked to this knowledge. In reality, we rarely find knowledge that can be regarded as purely mechanical. Nearly any abstract knowledge, even if stored as a memory record, has a certain link with other pieces of knowledge and therefore the student "understands" it in a certain way. It is possible to say that the measure of understanding abstract knowledge is determined by the richness and quality of generic models to which the abstract knowledge is linked.

Theory of Reification

Even though the aim of this study is not to compare TGM with other theories of knowledge acquisition, one of them is so close to our ideas that it is worth mentioning. It is the theory of reification (or TOR) (Sfard, 1991). The structure of TOR is depicted in the following scheme:

processes on objects→interiorization→condensation→reification→new object (ii)

The scheme is close in appearance to scheme (i). In both cases, gaining concrete experiences is an entrance to the process. In TGM, the emphasis is put on experience, in TOR on process. Other two terms in scheme (i) distinguish the process, i.e. generalization, and the concept, i.e. the generic model. In (ii), both these terms are procedural. Condensation corresponds to the term of generic model in the following way: The generic model is perceived as a mental structure built by generalization. The condensation is perceived as a period of "squeezing" lengthy sequences of operations into more manageable units. In other words, the condensation represents the way in which the generic model works, but it does not explain how it has been created. In TGM, the mechanism of its creation is described by congruent and contradictory references and grouping process. The abstraction process in (i) is also near to reification in (ii). The reification "is defined as an ontological shift – a sudden ability to see something familiar in a totally new light… reification is an instantaneous quantum leap: a process solidifies into object, into a static structure" (pp. 19-20). The reification is understood as sudden cognition. Similarly in Hejný (1988, p. 66), the abstraction was understood as "a very short period … in which …very strong emotion of delight is experienced, …". However, in the contemporary TGM the abstraction is frequently understood as a long-term process. We can say that the abstraction in (i) is created by a series of reifications in (ii). For example, the abstraction which yields the discovery of Pick's formula is frequently realised through a sequence of "smaller" abstractions revealing particular formulas.

GENERIC MODEL AND TEACHING STYLE

A transmissive teaching method neither takes countenance of the student's need to build the generic model individually nor appreciates the importance of the generic model. This can be seen in the following illustration.

Illustration 1. Alan (aged 16) was standing in front of the blackboard. He was to solve the following problem: From the team of 21 hockey players, the coach should choose a group of 5 hockey players for a penalty shot. In how many different ways can the coach choose a group of 5 hockey players? (A stands for Alan, T stands for the teacher.)

1 A: (The boy is reading the problem quietly.)
2 T: *So, what is it, Alan? Permutation or variation or combination?*
3 A: *It is like with the horses.*
4 T: (She is coming to the blackboard and she would like to say something.)
5 A: (He is writing down very quickly: $\binom{21}{5}$.) *It is twenty-one above five.*
6 T: *So you do not understand it much, but at least count it.*
7 A: (He is writing down: $\frac{21.20.19.18.17}{5.4.3.2.1}$, after some time of counting with the help of
8 a calculator, he writes down the result: 20 349.) *It is …*
9 T: *Ehm, good. But you have not said whether you have used permutation or*
10 *variation or combination.*
11 A: (hesitates) *Variation.*
12 T: *You have not written that. You have used combination. You've used a*
13 *formula and you do not know what it means.*

<u>Commentary to Alan's solution.</u> Alan solves the problem without any mistakes. In (3), he even explains the strategy he uses. He finds out that this problem is morphic to a problem which he solved earlier with much energy and time. The problem concerned the number of possible distributions of four horses into seven boxes in a stable. He grasps the organizing principle of the situation and understands how the problem differs from the case where four horses were considered as individuals. Thus, the horse situation serves as the generic model for the given combinatorial situation. Alan considers the word "combination" and other similar words as unimportant. His solving strategy will be called a *generic model solving strategy.*

It is worth asking how Alan's generic model is created. We have no direct evidence of this process but on the basis of several other processes of this type analyzed earlier, we can say that it is highly probable that the model is created in three phases. First by manipulating representations of horses and stables, Alan creates his own way of getting an insight into the set of all possible distributions. This process is based on the idea of decomposition of all possible distributions into several subsets. Second, the cardinality of each of these subsets is found and the numbers are added (in our case, it is probably 20+10+4+1 = 35). Third, this result is linked to the previous knowledge of combinatorial numbers $\binom{n}{m}$, in our case $\binom{7}{4}$. This concept, better to say pre-concept, is understood by Alan by means of the generic model "horses-stables".

The last phase might be reached in different ways. In our experimental teaching, the most successful way of guiding students to discover the formula $\binom{n}{m}$ was by solving the Abracadabra situations (Polya, 1966, p. 68). From the point of view of the procept theory (Gray, Tall, 1994), the first two phases might be regarded as a process, the generic model as a concept and the formula $\binom{n}{m}$ as a procept (provided it was not learnt by heart).

<u>Commentary to the teacher's reaction.</u> For the teacher, the generic model solving strategy is not sufficient. For her, the expected (and the only acceptable) solving strategy is based on the following process:

1. A given problem is classified and labelled. (It is clear that she expects Alan to categorise the problem as permutation or variation or combination.)
2. Using the label, a student recalls a procedure or formula, which is a base of the solving process. (Alan finds the appropriate formula (5), however not through the appropriate label (9).)
3. The student writes the procedure or formula generally and then substitutes concrete numbers from the problem. The teacher may have expected Alan to say:

> "It is the case of combination without repetition and the formula to solve the problem is $\binom{n}{m}$. It is the number of all m-tuples chosen from among the set of n objects. In our case $n = 21$ and $m = 5$." (*)

4. The final step is most frequently a calculation or realization of the procedure.

We will call the above solving strategy *label based solving strategy.* According to our experience, it is the most frequent cause of students' mechanical knowledge. From the experiments, many observed lessons and discussions with teachers, we know that many teachers believe that teaching the label based solving strategy is the most effective way to prepare students for entrance examinations to a higher level of schools. We notice that the teacher from the above illustration considers other solving strategies, especially the strategy of generic model, as insufficient (see (6), (9), (13)).

GENERIC MODEL AS A TOOL FOR A TEACHING SCENARIO

<u>Illustration 2.</u> Record of a lesson in Grade 6 (age 12-13) . (Note that the presence of a video camera, two researchers and several tape recorders in the classroom did not have a bad effect on the students' work, it motivated them to be more active.)

The class teacher uses constructivist approaches in her teaching. She does not present students with ready-made knowledge but by a suitable series of graded tasks, she directs them towards an autonomous discovery of knowledge. She knows that the generic model plays the decisive role in this process. Her teaching objective for the next two lessons is the discovery of the following abstract knowledge: A natural number n is divisible by 4 if and only if the last 2-digit number of n is divisible by 4. In brief,

$4|n \iff 4|$ the last 2-digit number of n. (iii)

She believes that some of the students will discover a similar rule for divisibility by 8. The lesson proceeds in the following way.

1. The teacher presents the students with a series of 4-digit numbers and challenges them into dividing the numbers into two boxes. The numbers divisible by 4 belong to the first box and the numbers not divisible by 4 belong to the second box. Then the

students are asked to find a simple rule, which differentiates the numbers in both boxes.

2. The information “the last digit must be even” gets around very quickly.

3. Michael says: “The number is divisible by 4 if its last digit is 4 or 8.” His statement is supported by the fact that numbers 1 524, 1 528 belong to the first box and numbers 1 522, 1 526 to the second box.

4. Jane shows number 1 520 as a counter example. Michael extends his rule by digit 0 on the unit place. Several students protest against Michael, some of them state other hypotheses. The teacher gives Richard an opportunity to express his hypothesis. He shows numbers 7 334 and 7 338 as counter examples to Michael’s hypothesis and says: “You cannot say that just by looking at the last digit.”

5. The camera records Dirk’s notes because Dirk does not get an opportunity to say his hypothesis. His list of numbers comprises about ten numbers in each box. Numbers 1 260, 1 280 (in the first box) and 1 230, 1 250 (in the second box) are underlined. The camera observes Dirk checking numbers 1 200, 1 210, 1 220, 1 230, 1 240, 1 250, written in the column. He crosses numbers 1 210, 1 230, 1 250 and sends numbers 1 200, 1 220, 1 240 to the first box using arrows. He writes “C is even”.

6. The teacher can see that the camera records Dirk’s exercise book and therefore gives him an opportunity to say his idea to the whole class. Dirk says: “Number 1 2C0 is divisible by 4 if and only if C is even. If it is odd, then it is not divisible.” He writes on the blackboard: “12C0 divis. 4 <=> C is even”. The teacher asks: “Is it correct?” Nobody reacts. Some students run to the blackboard to write their hypotheses. There are (besides Dirk’s) about ten records on the blackboard. E.g.:

Norbert: 1 212, 2 424 are divisible and all such numbers (he means the numbers for which A = C and B = D).
Kate: 2 222, 6 666 are not divisible, 4 444, 8 888 are divisible.
Lucy: 1 000, 2 000, 3 000, 4 000, …, 9 000, all these are.
Jane: 23odd2 is divisible.

7. The blackboard is crowded and the activity of the class grows into spontaneous discussions within small groups. About three students want the teacher to assess their solution. She encourages them to ask their classmates. No student asks the teacher to reveal the rule.

8. The teacher looks at the clock and stops discussions. She praises all students for their ideas and says: “I have not expected at all that you find so many promising ways how to find the rule for divisibility by 4. Because I have promised to our guests that we will get very near to the discovery of the rule, I leave only two records on the blackboard. I think that these records will lead us to the discovery very quickly.”

9. Meanwhile, Dirk, Richard and Jane sit together and after a while, Jane says that they know the exact rule. Hesitating, the teacher lets Jane express her idea. Jane says:

"Such a number (she writes ABCD) is possible to divide by 4 in these two cases: if C is even then D = 0, 4, 8; if C is odd then D = 2, 6. Otherwise it is not divisible."

10. At the end of the lesson the teacher says: "We can formulate the result of your work as follows: 4-digit number ABCD is divisible by four if and only if 2-digit number CD is divisible by 4." Finally, she poses the following voluntary homework:
Task 1. Find out how to decide whether a 7-digit number is divisible by 4.
Task 2. Find out how to decide whether a 5-digit number is divisible by 8.

11. The next day, Dirk and three more students bring their solutions to Task 2. All solutions are in the language of letters. For example, Dirk's solution is: *Number ABCDE is divisible by 8 in these four cases: 1. C is odd, D is even and E = 0, 4 or 8; 2. C is odd, D is odd and E = 2 or 6; 3. C is even, D is even and E = 2 or 6; 4. C is even, D is odd and E = 0, 4 or 8. In all other cases, number ABCDE is not divisible by 8.* The teacher is surprised that none of the four solutions uses the language of abstract knowledge:
A number is divisible by 8 if and only if its last three-digit number is divisible by 8.

Commentary

1) At the beginning, the teacher decides to only work with 4-digit numbers ABCD.

2) The previous students' experience with the concept of even/odd numbers enables the first classification into the set of numbers: all odd numbers belong to the second box.

3) Michael, remembering the abstract rule for divisibility by 2, 5, 10, focuses his attention on the last digit and formulates the incorrect hypothesis on the level of abstract knowledge. This is supported by four numbers which do not play the role of isolated models but serve as a tool for verifying the hypothesis.

4) As a reaction to Michael's hypothesis, Jana, Richard and maybe other students too accept an idea that it is necessary to notice the last digit and possibly others as well. Some more hypothetic generic models are discovered, formulated and tested. Finding the generic model is almost always quicker in a class than individually.

5) Dirk discovers the importance of the last but one digit C and this hypothesis seems to be meaningful. His investigative process is the first part of generalization and the obtained pattern is the first part of future generic knowledge. Contrary to Michael's grouping, which consists of several randomly chosen numbers, Dirk's grouping is already systematic since he identifies congruent and contradictory references and obtains the knowledge "C is even" as a generic model of the sub-group (D = 0).

6) Dirk is the first who discovers the rule on the level of generic model. Moreover, he expresses it in a sophisticated way. All other solutions (on the blackboard) are made by investigating a certain set of numbers ABCD about which it is possible to say something general from the point of view of divisibility by 4. Norbert formulates an incorrect hypothesis. Neither Kate nor Lucy is on a promising track to find the rule. The density of Kate's set of models is poor; Lucy's set of models lacks contradictory

examples. In her discovery process, Jane reaches the same level as Dirk. Her presentation of the result using the word "odd" might be more understandable for her classmates than Dirk's language of letters.

7) The students' behaviour shows that the teaching style is constructivist. For the majority of the students, it is this constructivist climate that enables them to partly construct or at least interiorize their generic models.

8) The teacher usually leaves opened problems as homework. However, in this case she wants the video recorded lesson to have a "happy ending". She wants to lead the students quickly to the discovery.

9) The analysis of Dirk's, Richard's and Jane's written records shows that Richard discovered earlier that the digits A and B were not important. Jane and Dirk discovered that it was necessary to consider the numbers with C odd and the numbers with C even separately. It is possible to say that the general rule was created as an amalgam of the three particular generic rules.

10) The teacher makes a didactic mistake changing Jane's objective generic model into abstract knowledge by means of translating the result into another language. Jane's generic model is in the language of concrete digits, which represents her object thinking. The rule formulated by the teacher is not linked to separate digits and has a character of abstract knowledge. Unlike the students' generic models, the teacher's abstract rule can be easily generalized to

- the rule of divisibility by 4 for more than 4-digit numbers
- the rule of divisibility by 8.

This is what the teacher anticipated when setting the two tasks as homework.

11) It is noteworthy that none of the students formulated their discovery in the 'elegant' language offered by the teacher. According to our understanding of the whole process, the reason why the students do not accept the teacher's suggestion of the abstract formulation is that they did not pass through the abstraction process on their own; the final abstraction was not interiorized. In terms of TOR, the absence of condensation was an obstacle for reification.

CONSEQUENCES AND REMARKS

1. In general, the generic model is more effective if dealing with concrete cases (former isolated models). For example, if Dirk (Illustration 2) has to decide whether 74 364 is divisible by 8, his rule is more effective than the abstract knowledge. However, if new knowledge is to be embodied into the whole mathematical structure, the abstract form of knowledge is more effective. It is also more effective if the given knowledge is to be generalized (into divisibility by 4, by 8, by 16) or transferred to another situation (divisibility in other than the decimal number system).

2. From our experiments, it follows that a student who is familiar with abstract knowledge often prefers the generic model when solving a task. Thus, we think that

Alan (Illustration 1) would have preferred the generic model "horses – stables" even if he had known (*).

3. Professional mathematicians often believe that the appearance of abstract knowledge strongly diminishes the importance of its generic model(s). The same belief can often be found in practising teachers, namely secondary school teachers. An example is the teacher in Illustration 1.

FURTHER ILLUSTRATIONS

The above illustrations cover just part of situations in which THE generic model plays an important or even crucial role. The following illustrations illuminate the situation further.

Illustration 3. Peter (aged 11) had to draw a square ABCD. The points A (0,0) and B (2,1) were given on a grid board. He found C (2,3), D (0,2) and called this quadrilateral a square. No one objected. This misconception is caused by a lack of generic models of squares drawn in a "skew" position. The concept of square is closely linked to horizontal and vertical directions.

Illustration 4. George (aged 13) had to find the area of triangle ABC where $|BC| = 5$ cm, $|AC| = 4$ cm and $|\angle ACB| = 120°$. He drew the correct figure with segment BC in the horizontal position. Then he wrote the formula $A = (a\ x\ h)/2$ and put $a = 5$ cm. He was unable to find the altitude of this triangle since he had never seen the altitude of a triangle which lies outside the figure. The restricted generic model of the concept 'altitude of triangle' is an obstacle for George's ability to find the solution.

Illustration 5. Lisa (aged 12) added 1.20 + 1/2 as 1.50. The seemingly meaningless result can be understood via determining Lisa's generic model of fractions. She used the clock face and interpreted 1.20 as 1h 20min and ½ as 30min.

SUMMARY

The generic model as the pivot term of our knowledge development theory was described in general and then used for the analysis of at least six following issues:

1. Teaching style. A teacher for whom the main goal of opening the mathematical world to students is the construction of the generic model, acts in a constructivist way. However, the teacher who transmits abstract knowledge to students, acts in a transmissive way. We saw both cases in Illustrations 1 and 2.

2. Creation of a teaching scenario (of a particular mathematical topic). The backbone of this scenario lies in the discovery of generic model(s). Frequently it is the case that different students construct different generic models, some of them simple and some of them sophisticated. This enables the teacher to address each student individually and in addition the rich spectrum of identified generic models allows the class to develop convincing generic model(s).

3. Student's discovery of relation. This process is based on building a sufficiently rich variety of isolated models. If this variety is poor, the corresponding generic model has a restricted application (see Illustrations 1 and 2).

4. Student's concept development process. The generic model usually serves as a pre-concept in the concept development process. In fact, it is an inevitable stage in this development (for Alan, "horses-stables" is the generic model of the concept $\binom{n}{m}$ and for Lisa, the clock face is an environment for the generic model of fractions).

5. Student's non-standard solving process. The kernel of a student's solving process is his/her understanding of the whole situation. He/she uses the previous experience generalised into the generic model and using morphism, he/she solves the problem. This process cannot be understood without the knowledge of the used generic model (see Illustrations 1 and 2).

6. Insufficient generic models as a source for the student's failure. Very often in the classroom practice, some concepts are presented only in "standard" forms. E.g. whole numbers as natural numbers, rational numbers as fractions (or decimal numbers), a triangle as an acute-angled one, a rectangle as a quadrilateral with sides in the horizontal-vertical position, a cylinder as a rotated cylinder with axes of rotation in the vertical position, etc. The consequence is a frequent failure of students (see Illustrations 3, 4 and 5).

Note: In points 3 and 4 we focus on creating generic models. In point 6, we consider consequences of the generic model which is not sufficiently developed.

Our further investigation of the generic model is aimed at:

7. Diagnosis of the student's mathematical knowledge. In short, abstract knowledge without any linkage to the corresponding generic model is mechanical.

8. Changing mechanical knowledge into knowledge with understanding. In short, missing generic models must be constructed.

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CLASSIFICATION LEADING TO STRUCTURE*

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Abstract: *Tasks aimed at classifying a group of solids using only tactile perception make pupils consider solids in a different way than when visual perception is involved. Pieces of information about the attributes of solids, gained through tactile perception, come to the pupils' mind gradually and this is projected into their way of manipulating solids. Analysing both the observations of pupils' manipulations and their verbal communication enabled us to construct the process of them building their structure of geometrical knowledge. Some results of our research on learning about solids and structuring geometrical knowledge are presented in the paper.*

Keywords: tactile perception, mental models, classification, manipulation.

INTRODUCTION AND FRAMEWORK

In this paper we investigate how the pupils' process the structuring of existing geometrical knowledge, then create new knowledge by extending the existing structure or its restructuring, or construct new knowledge occurs when pupils are classifying a group of solids. Some phenomena are discussed, related to the building of structure identified in pupils' mathematical behaviour when solving a geometrical problem and in a case study the process of constructing and reconstructing structure, related to a set of 3-D geometrical objects, is described and analysed. To avoid misunderstanding and misinterpreting of pupils' verbal expressions, the task was devised for tactile manipulation with 3-D geometrical solids and pupils' verbal communication was used to clarify our propositions.

Hejný (2002, 2003) propounds the theory that knowledge is gained by experiences, which in the first instance are unconnected, then an experience makes the pupil suddenly see a connection amongst several of the previous experiences. This triggers linkages between the experiences to form a network and which could lead to a generalisation. He states that the Internal Mathematical Structure (IMS) "binds all these networks together and equips them with an organisation". This approach together with four important properties that govern structure defined by Gestalt psychology (Van Hiele, 1986, 28) reflects our understanding of structure.

The dynamically nested RBC model of abstraction (Schwartz, Hershkowitz & Dreyfus, 2002a, b) was found to be a useful tool to help us to analyse, understand and describe the pupils' process of structuring their geometrical knowledge in the context given by a task (see below). All three epistemic actions, namely Recognizing, Building-With and Constructing are present in the observed process and we found that this theory fits the outcomes of our research closely. The three epistemic actions

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which are the constituents of abstraction are (Schwarz, Hershkowitz & Dreyfus, 2002b, 83):

> *Constructing* is the central step of abstraction. It consists of assembling knowledge artefacts to produce a new knowledge structure to which the participants become acquainted. *Recognizing* a familiar mathematical structure occurs when a student realizes that the structure is inherent in a given mathematical situation. ... *Building-With* consists of combining existing artefacts in order to satisfy a goal such as solving a problem or justifying a statement.

The pupils in our research are not starting to build knowledge from a zero base, they have some knowledge about solids from their experiences in and out of school. The pupils construct and build-with their knowledge to make structures for each individual solid and later use these structures to find 'cross-solid' structures between certain solids, that is, to find solids which are connected by some common attribute(s).

We consider three different levels of quality of mental picture of a perceived solid, which reflect the extent to which the pupil is familiar with it. For this purpose we applied Vopěnka's approach to geometry. In his study, Vopěnka (1986) introduces the concept of a 'personality' of a geometrical object, and his approach enabled him to give a deep analysis of the genesis of geometrical thinking. In a simplified way a solid (geometrical object) is considered as a 'personality' for the individual if s/he can associate the image with the name of solid, can describe some of its attributes, is able to recall the solid on the basis of a verbal description and in different positions and sizes, can represent it by a model or drawing, can recognise solids which are in some way related to it and describe the relationship. The three levels of the quality of the mental picture of a perceived solid are: 1. the solid is a 'personality' for the pupil, 2. the solid is unknown to the pupil, however, the pupil perceives some relationship between the considered solid and another solid which is a 'personality' for him/her, 3. the solid is entirely new for the pupil (Jirotková, 2001).

Our long-term research started by Hejný and Jirotková in 1994 leads us to believe that the structure which pupils build up in their minds, related to solids through visual and tactile perception, depends mostly on their life experience. In general the primary school mathematics curriculum does not offer many activities leading to creating geometrical structure. The solids, which are usually introduced to pupils by the age of 10-11 years, are the cube, cuboid, pyramid, cone, cylinder and sphere. The pupils learn them individually without possible connections between them. These individual solids might be committed in pupils' long-term spatial memory as 'personalities'.

When there are only tactile perceptions of a solid, which is not a 'personality' for the pupil, they may be of a general nature at first such as big/small, rough/smooth. When the pupils are asked to communicate verbally about their perceptions or when they are asked to perform certain manipulative operations, they usually begin to feel the attributes of the solid randomly (Van Hiele, 1986). However, these attributes which are committed to the short-term memory are usually related to one solid and are memorised as individual pieces of information, with the gestalt of the solid remaining the dominant feature. In our previous research (Jirotková & Littler, 2003; Littler & Jirotková, 2004) we found that when the pupils were aware that they would have to communicate their perception (tactile and/or visual), they began to work more systematically with each solid, counting all the edges, faces and vertices, both to get a more analytical picture of the solid in their minds and to be able to describe the examined solid more precisely. The cube is no longer a solid with which they are familiar, know its name and possibly differentiate it from cuboids simply by its shape, but is categorised in the pupil's mind as having eight vertices (often termed corners), six faces which are all squares and so on. Other common solids would be categorised analogically.

In our previous research the pupils were asked to group a number of solids into two groups such that all the members of each group had a common attribute. We focused on the pupil's understanding of geometrical phenomena and their communication about them. We could not get evidence of the process of solution of the task, because the solids were hidden in an opaque bag. On the basis of pupils' communication and the results of the task we constructed two important mental processes – the Mechanism of tactile classification and the Mechanism of visual check of tactile classification (Jirotková, 2001; Littler & Jirotková, 2004). We applied these mechanisms in this current research to find how and what geometrical structures the pupils create when solving a similar task.

We also have observed at first hand the truth of Bell's (1993) assertion that "a fundamental fact about learned material is that richly connected bodies of knowledge are well retained; isolated elements are quickly lost".

METHODOLOGY

Our study is focused on the process of building an internal mathematical structure (IMS) which is not a directly observable mental activity, hence we had to devise experiments which would show whether and how the pupil is developing an IMS. The following task was used as the tool of our experiment:

> Sort the solids into two groups, using only tactile perception, so that at least all the shapes in one of the groups have a certain common property.

Materials: The following 13 solids were used: cube (CU), square based prism (SPR), rectangular prism (RPR), triangular prism (3PR), non-convex pentagonal prism (5PR), hexagonal prism (6PR), square based pyramid (SPY), truncated rectangular based pyramid (TPY), non-convex pentagonal based pyramid (5PY), tetrahedron (TH), cone (CO), truncated cone (TCO), cylinder (CY).

Scenario: The pupil was asked to sit down in front of the screen behind which there were 13 solids. There were arm-holes in the screen through which the pupils could manipulate the solids but could not see them. The task was given to the pupil and their understanding of it was verified. S/He was then asked to put their arms through the holes and to carry out the task.

When the pupil indicated they had completed the task, pictures of the groups were taken and s/he was asked what criterion they had used to separate the solids into two groups. The groups were then taken from behind the screen so that the pupils could visually perceive them. They were asked whether or not they wished to change their selection in any way and if so why. Several additional questions relating to the solids and their relationships were asked at this stage to identify the level of the pupil's mathematical vocabulary and communicative ability.

The experiment was carried out in June 2004 with nine 10-11 years old pupils, three boys and two girls from an inner city school in the UK, three boys and one girl also from an inner city school in CZ.

The pupils' manipulation of the solids behind the screen as well as their communication was video-recorded, transcribed into a protocol form, and analysed first individually by each of authors then collectively with respect to phenomena related to structure and the epistemic actions forming RBC model in the process of classification.

Analysis of the task

The solids used in this task were deliberately chosen to cover three types of solids: 1. those the pupil would meet in school as part of the geometrical syllabus or in everyday life such as CU, SPR, RPR, CO, CY, TH and SPY; 2. more unusual solids such as 3PR, 6PR, TCO and TPY which might be known in some form from everyday life; and 3. two solids namely those with re-entrant angles, 5PY and 5PR, which we considered the pupils would not have met.

The basic set on which the task is constructed is the Cartesian product of the set of solids and their attributes. The task challenges the pupils to make a structure, a subset of the Cartesian product.

Pupils' geometrical knowledge is based on their real life experiences which differ from pupil to pupil therefore we expected to meet a variety of reactions and different solutions. Two types of classifying the solids into two groups were anticipated (Jirotková, 2001; Jirotková & Littler, 2002). The first type is *complementary classification* in which the pupil finds an attribute A, common to all the solids in one group and the other group is simply its complement, characterised by not having attribute A. The second type of classification is *attributal* in which each of the two groups is described by a certain property.

Visually one can get immediate information about the individual shapes of a set of solids and their mutual relationships. In this task, the pupil is familiarising him/herself with a group of solids using tactile perception only, perceiving the solids one by one. If comparison is needed, only a small group of solids can be perceived at any one time, which is why intensive collaboration with the short-term spatial memory is required. Tactile perception gives information on which the mental image of a shape is created. How quickly and how precisely the image is created depends to a certain extent on what geometrical phenomena are committed in the pupil's long-term memory. It could be for example a set of solids like cuboids, pyramids or some attributes of solids like regularity, non-convexity. Some geometrical phenomena could be associated with tactile perception like a prick, sharpness, smoothness etc. A particular geometrical phenomenon, the image of a solid, can be considered as a 'personality' (Vopěnka, 1986).

Data

The table below indicates the solids finally chosen by the pupils to be in their selected group. For each pupil, UK1-5, CZ1-4, the first line shows his/her tactile classification (T) into two groups, one is marked with # and one is blank. The second line shows the pupils' choice when they had visual perception (V) following the tactile classification. These two groups are marked with * and blank. Six pupils described the group marked with # or * by a common attribute – complementary classification. Pupils UK4, CZ3 and CZ4 described both groups – attributal classification.

		CU	SPR	RPR	TPY	3PR	5PR	6PR	SPY	5PY	TH	CO	TCO	CY
UK1	**T** **V**	# *	# *	# *	# *	# *	# *	 *	 *					
UK2	**T** **V**	# *	# *	# *	#	#			#					
UK3	**T** **V**	# *	# *	# *	# *	# *	 *		# *					
UK4	**T** **V**	# *	# *	# *	# *	# *	# *	# *	# *					
UK5	**T** **V**	# *	# *	# *	# *									
CZ1	**T** **V**	# *	# *	# *	# *									
CZ2	**T** **V**						# *			# *				
CZ3	**T** **V**	# *	# *	# *	# *	# *	# *	# *	# *	# *	# *			
CZ4	**T** **V**	# *	# *	# *	# *	# *	# *	# *	# *	# *	# *			

The pupils' verbal criteria for their tactile classification were as follows:

UK1: *They all have a square base, not triangular.*

UK2: *I put six on one side and the rest on the other*. (He first selected according to a criterion, but he forgot this when confronted with solids he did not know and placed the rest randomly). When given visual sight of the solids he changed his criteria to *'Cube or cuboid'*.

UK3: *This group has got square and rectangular bases and this group has got other bases*.

UK4: *This group has got rectangular or square faces at least. This group has got triangular or circular faces.*

UK5: *My group has only got quadrilateral faces. The other group may have quadrilateral faces but they have other faces as well.*

CZ1: *They have eight vertices. But they cannot be only squares. They can be any quadrilateral. I put aside all which were squares, rectangles, cubes and cuboids, then I added another shape which was like a not successfully made pyramid as if something was chopped off. The others I put on the other side.*

CZ2: *These solids had a piece bitten out from them.*

CZ3: *I put all rounded on one side and edged on the other side.*

CZ4: *If it was rounded or if it had a circle I put it here, and if it had edges then I put it in the other group.* She created the group shown above but described the complementary group because it was easier to do so.

DISCUSSION

First, we will discuss some phenomena derived from the table above, pupils' statements and our observations, and then we will focus on a particular pupil and analyse his process of classification in more detail. Our considerations are supported by findings from our previous experiments and the given table provides illustrations of some of them.

Cognitive phenomena

We have identified the following cognitive phenomena related to the process of structuring.

'Four-sidedness' as a criterion for complementary classification. From the verbal descriptions above it can be seen that the presence of squares and rectangles on a solid was perceived as a dominant feature of certain solids in most of the pupils' minds and thus it became an important structure making element (see UK1-5, CZ1). After considerable manipulation of many solids the same six pupils added the truncated rectangular based pyramid to the group of solids in which only cubes and cuboids were present.

A 'personality'. The cube, square based and rectangular based prisms were picked out by both the UK and CZ pupils and we observed that very little tactile perception was necessary. We consider this was because these solids were most familiar to the pupils from experiences both in and out of school and they were able to quickly bring a mental image from their long-term spatial memory to match that they had got from tactile perception. These three solids were most likely personalities for the pupils.

Global perceptions as selection criteria. We know from our previous research that both 'edgeness' and 'roundness' are easily tactilely perceived and these global attributes are often used as criteria for attributal classification (see CZ3, 4). Quickly after selecting the first three shapes (CU, SPR, RPR) many of the pupils then selected the cone, truncated cone and cylinder and put these in a separate group. Similarly 'non-convexity' was strong tactile sensation and became for the pupil CZ2 a criterion for complementary classification.

Confrontation of tactile and visual perception. It may happen that tactile and visual perceptions create different images about a solid. We identified this phenomenon when pupils wanted to change their tactile classification after visually checking it (see UK1-3). When the pupil UK1 perceived the hexagonal prism tactilely, a feeling of 'roundness' dominated, when she perceived the square based pyramid 'pointedness' or 'triangularity' dominated. When she perceived these two solids visually, she noticed the presence of a rectangle or a square and replaced them both in the group of 'four-sided' solids. For the same reason, the pupil UK3 added the non-convex pentagonal prism to the 'four-sided' solids. The sharpness of the edges was the initial dominant tactile sensation. On the other hand, she kept the hexagonal prism in the 'non four-sided' group after visual perception. This was because her thinking was dominated by base (see UK3) and the solid was 'sitting on' a hexagon. We can also see that the visual perception provides quite precise information about the measures of solids contrary to tactile perception. This is our explanation why the pupil UK2 separated solids TPY, 3PR and SPY from the group of cube and both cuboids after visually perceiving 'right-angledness'.

Conflict between attributal groups. Several pupils considered the triangular prism for a long time, feeling the triangular ends and the sharpness caused by the edges joining the 45^{o} vertices. One reason for their hesitancy into which group it should be put was that its faces were rectangles and triangles and several pupils were separating their solids into those which had rectangles and squares as faces and the second group had circles and triangles. Hence when the solid had attributes which fitted into both groups, a tension was created when deciding into which group to put the solid. Six pupils finally put it into the rectangle and squares group. The square based pyramid fell into this category too, having a square and four triangles as its faces.

Process of tactile classification – case of John (CZ1)

We now cite a case study of pupil John (CZ1). We chose John because, first he took the longest time over the task (approximately 3 times longer than the others), second, he started by attributal classification and then after creating four groups had to give it up and change to complementary classification. In the case study, the description of our observations, taken directly from a video-recording, is written in italics and split into stages (S1–S7) applying RBC theory. However, we are aware that the same task can lead to 'building with' or 'construction' depending on the pupil's knowledge, so our interpretations are necessarily subjective (Schwarz et al, 2002b). We make comments (C1–C7) on these stages, which include our interpretations of the pupils' actions.

S1. ***Looking for criterion***
John took the solids one by one as they came to hand. Some of them he tried to grasp at once, some of them he perceived as a whole briefly, some of them he manipulated carefully, moved them in his palm, touched their vertices with his fingers.
C1. John's manipulations with the solids indicate clearly which solid was a 'personality' in his mind – to those he paid little attention, but the solids which were new to him were observed for a considerable time, during which he tried to perceive their 'anatomy'. His initial familiarisation of the solids took 1.5 minutes and whilst doing this he tried to determine some criterion for classification.

S2. ***Creating the first structure, attributal classification – building-with***
*He took the non-convex pyramid (5PY) and put it into the group on his left (**L**). Then he put the cuboid (SPR) to the right (**R**) and after some manipulation he put the rectangular prism (RPR) in **R**. He then put both cuboids one above the other.*
C2. John first chose 5PY. This caused a distinct tactile sensation which recalled the image of a pyramid and its characteristic, 'pointedness'. This was the first selection criterion for group **L**. When he chose SPR, it became the carrier of the characteristic for the second group **R**. RPR was put into this group and then by placing the two solids one above the other, he expressed what these solids had in common. The type of classification at this stage was attributal, pointed solids (**L**) and cuboids (**R**).

S3. ***Need for restructure – construction***
*John perceived four solids at once (CU, 5PR, TH, 3PR), he then checked both groups **R** and **L**, each by one hand and then he returned to the four solids taking two solids in each hand. He put these down and tried to perceive how the groups of solids were arranged on the table. He took the non-convex prism (5PR) but seemed not to know where to put it. Then he took the 5PY in one hand and compared it with 5PR, which he held permanently, by careful touching. Quite clearly he perceived the vertices of the 5PR and he paid special attention to the vertices of non-convex angles. Finally he touched the vertices of 5PY with the right hand and after marked perception of its apex he put 5PY and 5PR into **L**.*
C3. When John perceived 5PY, which was a new solid for him, he perceived its gestalt and its dominant characteristic (pointedness) was put into his short-term memory. When he perceived 5PR he recognised that it was not possible to put it to the group **R** together with SPR and RPR and started to compare it with 5PY. However, there was the 'pointedness' characteristic in his short-term memory which did not allow him to put these two solids together in **L** immediately. In other words he could not put 5PR into existing structure. He wished to compare these two solids simultaneously so he had to use an 'external memory', that is he held a solid in each hand. Finally he realised they had a common attribute, non-convexity, so he put them together in **L**. We can derive from John's manipulation that he perceived this phenomenon as an analytical attribute of each solid which he perceived in a different way for each solid. For the 5PR he perceived two non-convex angles on opposite faces and for 5PY just one. His noticeable perception of the apex of 5PY indicated

that he was not very happy having to give up his first criterion, break his first structure and build up new structure in which 5PR fitted. The new structure building property was non-convexity.

S4. ***Using the new structure – building-with***
*John put the truncated cone (TCO) into **L**. He then added the tetrahedron (TH) to **L** after a check of all the shapes in this group. John then tried to hold 3PR, which lay on the table like a 'roof', it slipped from his grasp three times, then he carefully perceived the vertices by pairs on corresponding parallel faces. He put it to **L** but immediately took it out. After a new check of all the shapes in this group he put it back in **L**.*
C4. The classification of TCO and TH into **L** can be explained by his return to the original criterion 'pyramidity' which was recalled by 'pointedness'. His hesitation when manipulating with 3PR was probably caused by the domination of a rectangular face. The structure of the solids in **L** was not as firm as before because each solid was linked to 5PY but not linked to each other.

S5. ***Confirmation of structure – recognition***
*John put the cube (CU) directly into **R**, face to face with the SPR.*
C5. The cube was a 'personality' for him so he knew in which group to place it without any hesitation.

S6. ***Establishing a new structure – building-with and construction***
*He considered the hexagonal prism (6PR) for a long time, turning it, touching all the vertices of a face at the same time with his fingers. He put it back without making a decision, then picked it up again and put it in the **R** group for a while and finally placed it between the two groups **L** and **R**, creating a third and new group **N**.*
C6. It is clear that the solid 6PR was also new for the pupil and his tactile perceptions were not linked to such geometrical images which would enable him to classify it. As in the case of 3PR and later SPY, the perception of rectangular/square faces dominated, which led the pupil to classify 6PR first in group **R**. However this group is seen as strong in the pupil's mind and all solids are mutually linked by their attributes hence 6PR could not remain in group **R**. He could not put it into **L** because he did not succeed in linking 6PR with 5PY so he had to establish the new group **N**.

S7. ***Constructing new knowledge***
*He took 5PR again perceived all vertices carefully and placed it now into **N**. He hesitated where to put the cylinder (CY) and so replaced it on the table. Then he took SPY and after touching its base he placed it in **R** but immediately took it out and compared it with the cylinder and placed them both into **N**. He then took the TPY and investigated its attributes quickly, and placed it in **R**. Following this, he put all the shapes from **L** to **N**.*
C7. By placing TPY in **R** he restructured this group. The initially strong relationship was loosened from 'right-angledness' to 'having 8 vertices'. It seems that the newly created group **N** was linked only by the property 'cannot be placed into the group **R**' (see CZ1). The creation of this structure, no matter how weak it appears, was

accompanied by a careful investigation of the new solids. Moreover, group **N** was linked to **R** as being its complement. Thus his final classification is complementary which reverses his initial intention to classify the groups attributally.

CONCLUSIONS

This study is part of our long-term research on pupils' understanding of 3-D geometry investigating such aspects as cognitive abilities, spatial ability, communicative ability, visualisation and etc. We have used one of the tasks, developed previously, for this research to see whether it would enable the researchers to determine whether the task helps primary pupils to develop structures within geometry. The results showed that three types of structure were developed during the undertaking of the task:

- ***'single-solid structure'*** which is a solid and its attribute(s) (see structures $\mathbf{L}_1$=<{5PY}; pointedness> in S2 and $\mathbf{N}_1$=<{6PR}; cannot be put in **L** or **R**> in S6);
- ***'cross-solid structure'*** which is that structure which links several solids each having the same attribute (see final structure $\mathbf{R}_3$**=<**{SPR, RPR, CU, TPY}; have eight vertexes> in S7, which was developed from $\mathbf{R}_1$=<{SPR, RPR}; rectangularity> in S2, then $\mathbf{R}_2$=<{SPR, RPR, CU}; rectangularity> in S5 and finally $\mathbf{R}_3$ in S7, then $\mathbf{L}_2$**=<**{5PY, 5PR}; non-convexity> in S3);
- ***'web structure'*** which is several solids linked in pairs or small groups by some attributes but not to each other (see structures $\mathbf{L}_3$={5PR, 5PY, TCO, TH, 3PR} in S4, and $\mathbf{N}_2$ which is the complement to $\mathbf{R}_3$ in S7).

After completing the task the pupils realised they had now got a tool which would help them give the 'one-solid structure' to new solids and more importantly to look for 'cross-solid structures'. We feel it is important for teachers to know of these structure building processes and to use tasks such as the ones we advocate to develop this significant cognitive ability (building structures) in their pupils. We believe that we also showed that the theory of abstraction in context could be applied to the case of building geometrical knowledge on primary school level.

Our future research will be aimed at typologising the structure building processes and applying it to different contexts such as 2-D geometry and arithmetic.

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DISCUSSING THE CHALLENGE OF CATEGORISING MATHEMATICAL KNOWLEDGE IN MATHEMATICS RESEARCH SITUATIONS

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Abstract: *Starting with a quotation describing mathematical research, this paper presents ways of providing students with comparable experiences in mathematical research, in the classroom. The paper focuses on the benefits and implications for the students of such experiences. "Real mathematics research-situations" are defined, and the didactical goals of these situations, as they are experienced are elaborated on. These elements are presented through examples, looking at similar situations (research-situations) in two contexts and using different theoretical frameworks.*

Keywords: mathematical research situations, mathematical reasoning, experience of knowing, definition construction processes.

INTRODUCTION, THEORETICAL FRAMEWORK AND AIMS

It is widely accepted that school mathematics differs considerably, in scope as well as in purpose from what mathematicians do; the necessities of the classroom are certainly different from those of cutting edge scientific research. It is less obvious why the specific activities of students in the classroom are so different from what mathematicians do in their research. According to Corfield (2003, p. 35):

> Theorem proving, conjecturing and concept formation make up the three principal components of mathematical research. The brilliant observation of Lakatos […] was that these components are thoroughly interwoven. […] Mathematicians perform these activities simultaneously […]

In contrast, in the classroom, such three activities are marginalised to the extreme, so that students are rarely involved in research situations. We use naturally the word "research situations", but it has to be defined more precisely. We propose in this paper a characterisation of what we called "real mathematics research-situations". Of course, there are many such research-situations, and the ways we could approach them are also multiple. In this paper, we describe possible issues surrounding research situations by focusing on two sets of considerations: epistemological and social/didactical. These considerations stem from the following examinations:

Q1: why investigate research-situation?

Q2: what qualifies a research-situation as such?

Q3: what does it mean to "implement a research situation in the classroom" and what are the didactical goals? What can a student learn through research-situations?

In addition, to illustrate the implications of their implementation in the classroom, we call on two cases of such research-situations. These situations will be characterised in terms of the above-cited considerations as well as their individual contexts.

RESEARCH SITUATIONS: A MATHEMATICAL AND DIDACTICAL CHARACTERISATION

Mathematical (epistemological) characterisation

We call 'research-situations' situations of a mathematically open nature. This characteristic needs to be fulfilled for all parties engaged in the situation (students, teacher, professional researcher). In particular, such situations may be still open in the ongoing professional research. In addition, accessibility of such research-situations is established by insuring that the mathematical pre-requisites be of minor importance: anybody can engage in research-situations because it mobilizes only basic mathematical knowledge (of integers, or basic geometrical forms, etc.).

The situations are also characterised by a purposeful focusing on the engagement of students in a mathematical research process, and therefore, the emphasis is placed on the experience of process, as opposed to the acquisition of conceptual/technical 'content knowledge'.

An example of this can be found in the work of a 'Research Situations for the Classroom' (RSC) team called "Maths à Modeler" (http://mathsamodeler.net), centred on an initial presentation by a researcher for classes at different levels (from primary school to university). The study of these RSC suggests that they differ from what is traditionally known as problem solving by several characteristics:

- Questions are raised in a similar way to the approach of open conjectures in ongoing mathematical research and, sometimes, they are 'open' questions;
- the mathematical objects considered are not necessarily part of the explicit school curriculum, and the questions are generally not given in a mathematical form;
- there need not exist a unique answer (or any answer at all);
- a solved question can possibly lead to other new questions;
- the knowledge involved is more often 'transversal', such as arguing, conjecturing, proving, modelling, defining… bringing us back to Corfield's view.

This reflection about RSC is based on the notion that a researcher can, and often must, select his own suitable framework of resolution, modify the rules or redefine objects or questions. This is precisely the type of practice, fundamental to mathematical activity, in which we aim to involve students despite the fact that this type of practice is not frequent in class, and even seems practically taboo in many circumstances (Godot & Grenier, 2004).

The concept of 'transversal knowledge' cited in the list above has not yet been formally defined. At present, it refers to the skills and knowledge which straddle various mathematical domains and are used in a whole variety of mathematical contexts. In that respect, it relates to what Bruner defined as ‘non-specific transfer or, more accurately, the transfer of principles and attitudes’ (Bruner, 1960, p. 17). Indeed, transversal knowledge and skills allow the knower (student) to navigate within different mathematical domains. They are therefore more valuable as they are less context-bound. In this context, they include proving, conjecturing, refuting, creating, modelling, reasoning by induction or by decomposition/recomposition, extending but also transforming a questioning process, reasoning non-linearly, building definitions and having a scientific responsibility.

Didactical characterisation

It is essential, for these epistemological characteristics of the experience to be fulfilled, that the teacher takes a specific position, similar to that of a researcher faced with an open problem, and comprising an awareness of the involved transversal knowledge (Godot & Grenier, 2004). This brings us to the social and didactical contract (Brousseau, 1997) that will produce the appropriate context for these activities to lead to the desired goals. There are many ways to grasp and characterise a research process: a didactical viewpoint is proposed in Godot & Grenier (2004) for instance, and Rota, in his introduction to *The Mathematical Experience*, explains that:

> A mathematician's work is mostly a tangle of guesswork, analogy, wishful thinking and frustration, and proof, far from being the core of discovery, is more often than not a way of making sure our minds are not playing tricks. (Davis & Hersch, 1981, p. xviii)

The didactical contract needs to focus on the aspects of mathematics research which are relevant and the way these aspects can be made present in the experiment. To illustrate, Rota’s description suggests that the creative process in mathematics research is a messy activity with no guarantee of successful results. This contrasts with the traditional classroom experience where the students seek a definite solution that the teacher already knows. Each aspect such as this one needs to be evaluated for usefulness, then, if possible, reformulated for the classroom context.

Many of the teacher’s decisions impact on the format and content of the activities. Firstly, mathematical research is a creative endeavour, and cannot be easily framed into the occasional one-hour session. The timeline must therefore be made to reflect this characteristic: the student must have the opportunity to appropriate the experience of the process reflexively, and to pass the frustration point where the temptation to give up is the strongest.

Secondly, as discussed above, the mathematical concepts have to be accessible enough to the participants that they can focus on their research process (in a reflexive endeavour). This is designed to ensure that the responsibility and power is shifted to the participating students and it is accomplished in three ways. To begin with, the students are given ownership of the experience through their own process of formulation of the question/problem. In consequence, and secondly, the teacher does not know the answer in advance, or even if there is one. Thirdly, and most importantly, the teacher works as a sounding board only, in order to avoid leading the process.

Finally, these constraints need to be recombined with the needs of the curriculum, including the teaching of basic notional mathematics. The following two sections illustrate instances where this plan was implemented.

EXAMPLE 1: RESEARCH SITUATION AND EXPERIENCE OF KNOWING

In 2003, a class of elementary student teachers in an American university spent two months in one of their required mathematics courses on a research project 'at their own level' (Knoll et al., 2004). They spent a month investigating research situations similar to the one described above, in informal groups. In many cases, there was not even a specific question, let alone a unique answer. In the second month, the students chose one of the investigations and took it or one deriving from it to a deeper level. Remember, these are not subject specialists; despite that, each student or group of students conceived their own topic!

In this particular case, the students investigated geometry topics such as proper colourings[1], polyhedra, and tilings. Note again, that the mathematical objects were easily accessible. And of course these problems, though seemingly trivial to a mathematician, have not all actually been solved; their resolution would not increase the canon, because it does not require the development of new mathematical tools, and so they are left out, but that is another story.

Concerning the anticipated achievement of the participating students, the study focused on their relationship to the subject of mathematics, including attitudes, beliefs and practices. Indeed, if a knower sees mathematics as made of a closely interconnected network of concepts, skills and relationships, she will be more likely to operate at a higher level than if she regards it as an amalgam of disconnected facts and procedures. To illustrate, the theoretical framework is summarised into a table with the action of knowing ('modes of knowing') in one direction, and the object of the knowing ('notions') in the other (see Table 1, below).

In this model, both categories are divided further, creating a matrix describing various situations. The key to this categorisation is that *different people could place experiences of knowing the same mathematics in different cells*.

In addition, the columns are distinguished by whether their content is (a) reproducible, (b) transferable, or (c) reconstructible. Clearly, a notion is not known if it cannot be reproduced. Further, if you know *why* something is the case, chances are you would be able to use it in a different situation. For example, understanding that the 'carry the 1' action in two column addition comes from the notion of place value,

[1] A proper colouring is a colouring of a subdivided system such that no adjacent cells share the same colour.

can lead to being able to transfer this to the case of two column multiplication without being told.

Modes of knowing **Notions**	**Knowing that/how** *Reproducible but neither transferable nor reconstructible*	**Knowing why** *Reproducible and transferable but not reconstructible*	**Knowing when** *Reproducible and transferable AND reconstructible*
Convention arbitrarily chosen	Memorised information and use	Nothing to understand/derive	Cannot be reconstructed by reasoning
Application moving from theory to practice	Subjectively the same as a 'convention': memorised information and use	Derived from other notions using the logical structure of mathematics	Derived from other notions using the logical structure of mathematics
Theorisation moving from practice to theory	Subjectively the same as a 'convention': memorised information and use	Derived from other notions using logical structure of mathematics	Derived from other notions using logical structure of mathematics

Table 1: Ways of Knowing and Notions

In the case of 'reconstructible' knowledge, the knower has grasped the mathematical structure underlying the notion to such an extent that, given the need, she would be *able to reconstruct it*. This distinction is important in that it implies a deeper understanding of the mathematical structures from which the specific emerges, giving it more transferability potential and a more wide-ranging applicability, making it more fundamental and going back to the notion of transversal knowledge. This is not saying, however, that it applies only to higher domains of mathematics.

Let us now look at the rows, the categories of knowledge. There are many models for this in the literature on mathematics education (Piaget, 1970; Bell et al. 1983; Skemp, 1987, Hejný, 2003; etc.). They are generally constructed to emphasise one aspect or another, or to make key distinctions. For the sake of clarity, in the present case we will refer to the fragments of specific knowledge as notions, avoiding thus the need to specify to whose definition of 'concept', 'skill', etc. we are referring.

The first and perhaps most important distinction separates a convention from the others. This distinction is important in that it is carried through to the ways of knowing. As can be seen in the table, a 'convention' is not the result of a logical derivation from a more basic or fundamental entity. It is somewhat arbitrary. This is key: Considering a given notion as a convention is a kind of fallback position, when the learner just cannot grasp something. The learner will then regard the notion as something that was decided for reasons that remain obscure, or even arbitrarily determined by someone else and take it at face value. This mechanism can be the correct one, but mostly it will lead to problems.

The second distinction is between applications and theorisations. Both categories contain the results of mathematical reasoning, unlike conventions. In addition, the two categories are distinguished through the direction of activities that call on them. In the case of application, notions are used to solve problems, to execute algorithms, and perform other activities that take the knower from the general, abstract, theoretical, to the specific, concrete, applied, as in the majority of traditional classroom work.

In contrast, the theorisation category operates from the specific, concrete, applied case to the general, abstract, theoretical. This is what is used in the mathematical activities described earlier: proving, conjecturing, refuting, defining, etc.

Evidently, the right end and the bottom of the table represent deeper thinking. In addition, the whole network is interconnected. Unfortunately, in many models, the distinction between the centre and right columns is left out and the two are collapsed or even left out altogether. This deeper thinking, which relates to the transversal knowledge described above, is therefore little emphasised, *or verified and assessed* in conventional classroom activities, even though it is much more fundamental and most importantly more resilient.

The important point to consider in this model is that moving towards the right *does not imply* delving into higher mathematics. The level of mathematics constitutes a third, independent dimension, and theorisation notions can be accessed in the context of very accessible mathematics, as indeed can 'knowing when' be achieved.

The project formed an experiment focusing on this. In fact, the course was designed to direct the students' attention onto their engagement in a mathematical research process, as discussed earlier, with special attention to the elements on the right and lower ends of the table. This was done using a whole battery of strategies, including the use of writing and portfolios for assessment, reflections and other interactions with the didactician. This last in particular was encouraged through the use of reflective student journals and the emphasis of the final project report on the students' process as opposed to their results. The participants found generally that the atmosphere of the classroom was unlike the mathematics context they were used to. Several commented that their outlook if not their feelings had changed, and many realised that mathematics was more than they had previously been led to believe (Knoll et al., 2004).

EXAMPLE 2: RESEARCH SITUATION AND DEFINITION CONSTRUCTION PROCESSES

Processes of defining represent an important part of mathematical activity, as underlined by Lakatos with an example concerning the immersion of a proof in a classification task:

> *there are other ways of communicating meaning than definitions. I, for one, shall initiate my pupils into the problem-situation which I am dealing with not by definitions, but by showing them a cube, an octahedron and showing that for these V-E+F=2. Then I shall ask for the domain of validity of this formula. (Lakatos, 1961, p.69)*

In this context, Lakatos shows that a definition is not only a tool for communicating, but also a mathematical process taking part in the formation of concepts. In the example at hand, the aim consists of a characterisation of markers in order to examine the concept formation process, and, in particular, to identify specific statements in the defining processes in order to say something about concept formation.

If we consider classification tasks as a part of the definition building context, this definition building process itself takes place within the wider problem that is the search of a proof, which in turn catalyses the construction of the concept, of polyhedra for example (Lakatos, 1961). If we concentrate our attention on a classification situation as a definition construction situation (it can be a particular RSC), this may appear simple: one takes some examples and counter-examples of a mathematical object and asks for a definition. We have experimented with this type of defining situation with the mathematical object 'tree' (see Ouvrier-Buffet, 2003) and also with the mathematical object of the 'discrete straight line' (see Ouvrier-Buffet, 2004). Both these situations have been conducted with students in their first year of university (scientific and not scientific sections). These mathematical objects are noteworthy because they are accessible by their representations, and are non-institutionalised, thus no pre-existing definition of these concepts exists. Furthermore, let us notice that to classify is a familiar task, both in everyday life and in sciences (in geometry or in biology for instance). Moreover, some students' conceptions about mathematical definitions (what definitions are or should be, what they do, the aspect they should have, etc.) will certainly have a leading role in such a situation, and may represent an obstacle to the defining process, or even a catalyst. We have to be aware of this fact, but the main question stays: are students capable of awareness of their own defining process? We want to study this kind of reflexive process. So, the challenge is now to characterize defining processes and situations in which a definition has to be built.

There exists a model for defining processes (Ouvrier-Buffet, 2003) which emphasises the *operators* and *controls* taking part in the processes. Modelling defining processes involves exploring the procedures implicated in the creativity of professional research mathematicians when they build new concepts (admittedly this is no small challenge!

That is why the roots of the presently characterised operators and controls, which are taking part in a defining process, are epistemological and philosophical[2])

The study also involves the identification of the markers of these processes in order to analyse how students define. These markers also allow a first characterisation of some key tools at the teacher's disposal for a definition building activity. Let us present some element of this. Remember that the teacher, in an RSC, is *not* the holder of knowledge. He has to adopt the position of researcher, like the students. Even then, there are 'didactical levers' such as recalling the instructions and asking for a definition.

The study of the concept of definition shows that a defining process is based on four poles, graspable in three epistemological conceptions: one concerns the *construction of a theory* (Popper, 1963), another deals with *Problem-Situation* (Lakatos, 1961), and two other poles concern the *logical* and the *linguistic* aspects (Aristotle), respectively. In this context, the teacher (who becomes a **M**anager-**O**bserver in an RSC) may interfere in the defining process of the students through logical requests, linguistic or axiomatic exigencies or the supply of given counter-examples. The MO can also ask for the construction and/or recognition of an object (tree or discrete straight line in a classification task for instance): it is a request obviously related to the function of the definition (does the definition help to recognise or construct the discrete object?). For instance, a question like this: "draw a discrete straight line crossing these two given pixels" engages students in a new reflection, of an axiomatic kind; the uniqueness of such straight lines is of crucial importance, and implies thus a new movement in the defining process.

This last statement necessitates a wider characterisation of the teacher's levers for defining situations. It can be comfortable for the teacher, because this kind of activity (the didactical contract for students is to build a definition) leads to a product, re-usable in a course. The teacher also remains in control of the process to a large extent, because there are clear goal posts.

CONCLUSION/ OPENING REMARKS

This paper brings an overall picture of the potentialities of research situations for the classroom. RSC give us an opportunity to work on scientific processes, constituted by students' experiments with different cognitive attitudes: doubting, conjecturing, refuting (generating new counter-examples), testing etc. In particular, the processes of defining can be modelled through four main items: formulating, logic, heuristic,

[2] We will propose an integrated picture of these operators and controls, in relation with different kinds of defining situations during the conference, with a poster entitled "On Modelling Conceptions about Mathematical Definitions". We will also present an illustration of the use of this model with a definition-construction situation.

theorising. The situations also give potential for students' reflections on wider issues, for example about the nature of mathematics, or even knowledge in general.

The discussions initiated in this paper illustrate the challenges facing education researchers interested in the research-situations as applied in the classroom. Further work concerning the nature of transversal knowledge and the understanding of a mathematical concept is fundamental. We have now to continue the implementation of research situations in the classroom in order to refine the characterisation of transversal knowledge and to define "properly" the learning in a specific context: that of the creation of knowledge, both in the discipline in general and in the mind of the learners.

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ON LINGUISTIC ASPECTS OF STRUCTURE BUILDING

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Abstract: *On a historical case study we examine the connectedness between structures that exist on different levels of abstraction. We take the history of algebraic equations and following the development of algebraic symbolism we show how the maturation of one structure is a precondition of the emergence of the next one.*

One of the typical features of cognitive structure building is its hierarchical nature. Structures exist on many different levels of abstraction. Therefore besides horizontal connectedness, that is connectedness among elements on the same level of abstraction, also structures that exist on different levels of abstraction are connected. For example the notion of a group is connected not only with notions like the notion of a ring or the notion of a field, which are on the same level of abstraction as the notion of a group itself. The notion of a group is connected also with the notion of a quotient group, which represents the next higher level of abstraction on which operations with groups themselves are being performed. On the other side the notion of a group is connected also with permutations and symmetries, which form the lower level, that is the level, from which the notion of the group was abstracted.

An important problem of the theory of cognitive structure building is to understand how structures on different levels of abstraction are connected. One of the tools, which foster the coexistence and connectedness of structures on different levels of abstraction, is symbolic language. In the paper I would like to examine the construction and coexistence of three different layers of structures in algebra. For this purpose I will use historical material, because in history of mathematics we can follow the process of structure building over a long period of time and so examine, how the different layers of abstraction follow each other and how different layers influence each other. I will concentrate on the development of algebraic structures from Al Chwárizmi to Euler, thus just before the notion of an abstract algebraic structure appeared. This analysis reveals an interesting twofold dynamics of structure building:

1. On the one side there is the ***turning of a process into an object***. This part of the dynamic is well understood by the theories of concept formation (see Hiebert (1986), Sfard (1989), Gray-Tall (1994), Hejný (1999)).

2. There is also a dynamics in the opposite direction. After a process has been turned into an object, the new objects are expressed by symbols, and these symbols enable the ***emergence of a process on a higher level*** of abstraction, a process of manipulation with these symbols (see Corry 1996).

In order not to mix the two levels, the objects representing the processes of the first level must be rather rigid, the process of object formation must be stabilized. That is why the turn from the first to the second level took in history several decades. It is not clear whether similar phenomena could be observed also in the development of algebraic thinking of children. Thus there is an open question for experimental research, whether it is possible to find spontaneous building of one structure on the basis of another. But let us now turn to the historical case. We shall examine the language by the help of which algebraic equations were solved, and discriminate three layers of cognitive structure.

1. The solution of an equation as a rule

Abú Abdalláh Muhammad al-Chwárizmi (780-850) is the author of the *Short book of algebra and al-muqabala* a treatise on solving „equations". The word *al-gabr* (algebra) in the title of the book came in time to be used as a name for the whole discipline dealing with „equations". We cannot speak about equations in the modern sense, because the book of Al Chwárizmi ***makes no use of symbols*** and even numbers are expressed verbally. For the powers of the unknown the book uses special terms: for x it uses *shai* (thing), for x^2 – *mal* (property), for x^3 – *kab* (cube), for x^4 – *malmal*, for x^5 – *kabmal*, etc. Algebra was understood as a set of rules for manipulating with the thing (i.e. the unknown), which enable us to find the solutions of particular „equations".

Before attempting to solve an „equation", Al Chwárizmi first rewrote it in a form where only positive coefficients appeared and the coefficient of the leading term (term with the highest power of the unknown) was one. In order to achieve this standard form, he made use of three operations: ***al gabr***–if on one side of the „equation" there are members that have to be taken away, the corresponding amount is added to both sides; ***al-muqabala***–if the same power appears on both sides, the smaller member on the one side is subtracted from the greater one on the other side; and ***al-rad***–if the coefficient of the highest power is different from one, the whole „equation" is divided by it. We write the term „equation" in quotation marks, because Al Chwárizmi did not write any equations. Rather, he transformed relations among quantities, everything being stated in sentences of ordinary language, enriched by few technical terms. Piaget characterized structure by transformations (Hejný 2002, p. 15), thus *al gabr*, *al-muqabala*, and *al-rad* can be viewed as transformations creating the first layer of algebraic structure.

We can illustrate this structure with an example. Consider the equation $x^2 + 10x = 39$, which Al Chwárizmi expressed in the form: „*Property and ten things equals thirty nine*". His solution reads as follows: „*Take the half of the number of the things, that is five, and multiply it by itself, you obtain twenty five. Add this to thirty nine, you get sixty four. Take the square root, or eight and subtract from it one half of the number of things, which is five. The result, three, is the thing*". This is a set of specific instructions telling us how to find the solution. Nevertheless, Al Chwárizmi has ***the notion of the unknown*** (shai) and therefore his instruction „*take the half of* *the number of the things, multiply it by itself, add this to thirty nine, take the square root, and subtract from it one half of the number of things*" is a universal procedure, which can be applied to any quadratic equation of that particular form. Thus he is able to grasp the procedure of solution in its entire universality. When he uses concrete values for the coefficients, he does so only for the purpose of illustration. With the help of the notions as *shai*, *mal* and *kab* he is able to grasp the universal procedure, and in taking this step he became the founder of algebra.

2. The solution of an equation as a formula

In the 12th century the works of Al Chwárizmi were translated into Latin. The custom of formulating the solution of an „equation" in the form of a verbal rule persisted until the 16th century. The first result of western mathematics that surpassed the achievements of the Ancients was formulated in this way. This was the solution of the cubic equation, published in 1545 in the *Ars Magna Sive de Regulis Algebracis* by Girolamo Cardano (1501-1576). Cardano formulated the equation of the third degree in the form: „***De cubo & rebus aequalibus numero.***" The solution is given in the form of a rule: „*Cube one-third of the number of things; add to it the square of one-half of the number; and take the square root of the whole. You will duplicate this, and to one of the two you add one-half the number you have already squared and from the other you subtract one-half the same. You will then have* **binomium** *and its* ***apotome***. *Then subtracting the cube root of the apotome from the cube root of the binomium, that which is left is the thing.*"

In order to ***see*** what Cardano was ***doing***, we present the equation in modern form $x^3 + bx = c$ and we express its solution in modern symbolism:

$$x = \sqrt[3]{\frac{c}{2}+\sqrt{\left(\frac{c}{2}\right)^2+\left(\frac{b}{3}\right)^3}} - \sqrt[3]{-\frac{c}{2}+\sqrt{\left(\frac{c}{2}\right)^2+\left(\frac{b}{3}\right)^3}}\ .$$

Of course, Cardano never wrote such a formula. In his times there were no formulas at all. On the surface algebra was still *regula della cosa*, a system of verbal rules used to find the thing. Nevertheless, below this surface some fundamental changes were taking place.

Even if Cardano's rule itself did not deviate from the framework of Al Chwárizmi's approach to algebra, it is not clear how was it possible to discover something so complicated. In order to understand this, we have to go back a century before Cardano and describe the first stage of the reification of the language of algebra, connected with the ***creation of algebraic symbolism***. After western civilization had absorbed Arabic algebra, a tendency arose to turn algebraic operations into symbols. This process was slow, lasting nearly two centuries. We will present only some of the most important innovations. Regiomontanus (1436-1476) introduced the symbolic representation for root extraction. He denoted the operation of root extraction with the capital *R*, stemming from the Latin *radix*. Thus for instance he expressed the third root of eight in the form *R cubica de 8*. In this way he represented the ***operation*** of root extraction by the ***expression*** of the root itself, that is by the result of the operation. Michael Stifel (1487-1567) replaced the capital *R* by a small *r*, so that instead of *R cubica de 8* he wrote $\sqrt{c}8$. He introduced the convention to write the upper bar of the letter *r* a bit longer. Stifel placed the first letter of the word *cubica* below this prolonged bar, so that everybody would know that it was the cube root. The number placed after this sign is the one whose root is to be extracted. Our modern convention was introduced by Descartes (1594-1650). Descartes replaced Stifel's letter *c* by the upper index, and placed the number itself below the bar of the letter *r*, thus writing the third root of eight in the form $\sqrt[3]{8}$. This changing of the letter *c* into the numeral 3 opened up the possibility of devising arithmetical rules for handling exponents.

Another very important development took place in connection with the representation of the unknown. The Arabic terms of *shai*, *mal* and *kab* were translated as *res*, *zensus* and *cubus*. Instead of writing the whole words mathematicians started to use only their first letters, thus *r* for *res*, *z* for *zensus* and *c* for *cubus*. Just like the Arabs algebraists, the *Cosists* (as the practitioners of this new algebra were called) did not stop with the third power of the unknown, introducing higher powers, such as *zz* (*zenso di zensi*), *zc* (*zenso di cubo*), etc., and developing simple rules for calculating with such expressions. Through such gradual processes symbols for the algebraic operations were introduced and gradually a ***whole layer of operations was reified***, acts were turned into objects. (It would be interesting to compare the historical case with the theory of reification in maths education, see Sfard 1989.) This process was slow, and at the beginning it was only little more than replacing words by letters for the sake of brevity. When the new symbols accumulated in sufficient quantity, they made possible a radical change in algebraic thought–the solution of the cubic equation. As we saw, Cardano formulated his result as a verbal rule. Nevertheless, its discovery was made possible by the new symbolism. We will present a reconstruction of this discovery, presenting it in modern symbolism for the sake of comprehensibility (see Scholz, 1990).

Let us take a cubic equation

$$x^3 + bx = c,$$

The decisive step in the solution of this equation is the assumption that the result will have ***the form of*** the difference of two cube roots. We do not know how the Italian mathematicians hit upon this idea. When we make this assumption, everything becomes simple. Let

$$x = \sqrt[3]{u} - \sqrt[3]{v}\,. \qquad (1)$$

Raising this expression to the third power and then comparing it with the equation we obtain the following relations between the unknown quantities *u* and *v*, and the coefficients *b* and *c*:

$$b = 3\sqrt[3]{uv} \qquad\qquad c = u - v. \qquad (2)$$

When we isolate v from the second equation, and substitute the resulting expression into the first one, we obtain a quadratic equation

$$u^2 - uc - \left(\frac{b}{3}\right)^3 = 0. \qquad (3)$$

The root of this equation is given by the formula for quadratic equations as

$$u = \frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}. \qquad (4)$$

The value of the unknown v can be now determined from the second equation of (2). Knowing u and v we can find the solution of the original problem from (1)

$$x = \sqrt[3]{u} - \sqrt[3]{v} = \sqrt[3]{\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}} - \sqrt[3]{-\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}}.$$

In this derivation the advantage of the algebraic symbolism, i. e. of the reification of the language of the Arabic algebra, is clearly visible. Right at the beginning we assumed that the result would have ***the form of*** the difference of two cubic roots. In Arabic algebra there were no expressions, there were only rules. And *a rule has no form*, because it cannot be perceived. We can only listen to it, and then perform all the steps precisely as the rule instructs us. Only when we represent the steps of the rule by symbols does the sequence of calculations appear before our eyes, only then are we able to perceive its form. ***The rule is thus transformed into a formula***. Algebra becomes an *analytic art*, the art of transforming algebraic formulas, guessing the form of the result and finding suitable substitutions. This art forms the core of Cardano's *Ars Magna*.

Nevertheless, it is important to realize that new possibilities are opened up when one reifies one level of the algebraic language by turning the rules into formulas (i.e. when the rule „square the thing and add to it five things" becomes simply the „$x^2 + 5x$"). Al Chwárizmi knew only three algebraic operations–*al-gabr, al-muqabala* and *al-rad*–but not ***substitution***. His transformations did not make it possible to change the „form" of the algebraic formulas. Substitutions are used in order to simplify algebraic problems, transforming for instance a cubic equation for x into a quadratic equation for u. The substitutions do not simply shift a term as a whole from one side of the equation to the other but rather decompose them and then rearrange them in a new way. For instance, the substitution $x = \sqrt[3]{u} - \sqrt[3]{v}$ decomposed the unknown x into two parts, rearranged these parts, and then put them together again. Thus it seems that this transition ***shifts the ontological foundations*** one level deeper. Algebra as *regula della cosa* understood the unknown as a „thing", that we can „take in our hands and move to some other place", but the „form" of this thing remained unchanged. Now the „thing" itself is transformed. It is, for instance, decomposed into two parts, which can be treated separately.

Another important change introduced by the use of substitutions concerns what counts as the solution of an equation. Formerly, in the framework of algebra understood as *regula della cosa*, mathematicians accepted only positive solutions, because the number of the things cannot be negative. If the unknown represents some real quantity, some number of things, it cannot be less than nothing. But as soon as we start using auxiliary equations, the unknowns of which refer only indirectly, due to substitutions, it can happen that the positive solution of the original equation corresponds to a negative root of the auxiliary equation. Therefore in the auxiliary equations we have to take into account ***the positive as well as the negative roots***. In this context Cardano distinguished between the „true“ and the „false“ solutions. The notion of an equation is thus slowly freed from its dependence on direct reference.

The most important shortcoming of the previous algebraic notation was that it used different letters (*r*, *z*, *c*, ...) to represent the powers of the same quantity. Thus, for instance, if *r* is 7, then *z* must be 49, but this dependence is not indicated by the symbolism. When substitutions are used, such a convention becomes unwieldy, because whenever we make a substitution for *r*, we must also make the appropriate substitution for *z*. Further, in a substitution we have to do with at least two unknowns, the old one and the new one. To represent both with the same letter *r* would create an ambiguity. Another shortcoming of the symbolism of the *Cosists* was that it had no symbols for the coefficients of the equation. Instead they used such phrases as „the number of things“, meaning by this the coefficient of the first power of the unknown. The symbolism was unable to express the coefficients in a general way, and consequently they only used fixed values of the unknowns in their symbolic manipulations.

In 1591 François Viéte (1540-1603) published his *In Artem Analyticem Isagoge* (Introduction to the analytic art). In this book Viéte introduced the symbolical distinction between unknowns and parameters. He was the first to represent the coefficients of equations with letters. It is only beginning with his work that we can speak of a universal formula, which expresses the solutions of an equation in terms of its coefficients. Viéte used capital vowels *A*, *E*, *I*, *O*, *U*, to represent the ***unknowns*** and the capital consonants *B*, *C*, *D*, *F*, *G*, ... to represent the ***coefficients***. In addition, each quantity had a dimension: 1-*longitudo*, 2-*planum*, 3-*solidum*, 4-*plano-planum*, ... The dimension of each quantity was expressed by a word written after the symbol, thus for instance *A planum* was the second power of the unknown *A* (what we now write as x^2) while *A solidum* was the third power ***of the same unknown***. Thus the letter indicates the identity of the quantity while the word indicates its particular power. This expedient makes it possible to use more than one quantity, and among other things makes it possible to express a substitution. Even though Viéte’s symbolism is rather complicated, it was ***the first universal symbolic language for the manipulation of formulas***. Viéte was fully conscious of the importance of his discovery. He believed that this new universal method would make it possible to solve all problems.

3. The solution of an equation as a splitting of a form

One of Cardano's merits was the systematic nature of his work. Therefore besides the equation of the form „*cubus and thing equal number*", the solution of which was discussed above, he presented rules for the solution of the other two forms of cubic equations. The rules for the solution of these equations have a form very similar to the first case. Nevertheless, Cardano made a surprising discovery when he tried to apply his rule for the equations of the form „*cubus equals thing and number*" to the equation $\boldsymbol{x^3 = 7x + 6}$. When he applied the rule he obtained a result we would express as follows:

$$x = \sqrt[3]{3+\sqrt{-\frac{100}{27}}} + \sqrt[3]{3-\sqrt{-\frac{100}{27}}}\ .$$

Below the sign for the square root a negative number appeared. The formula required him to find $\sqrt{-\frac{100}{27}}$, something he was not able to do. For the further progress of algebra it was crucial to understand what was going on when a negative number appeared below the square root sign. The discovery of the *casus irreducibilis*, of the insoluble case, led to a gradual loosening of the bond between language and reality. The algebraic expressions are viewed more and more as ***forms***, as formal objects constructed from symbols, independent of any realistic context in which they are supposed to be interpreted. An important motive for such a development was the situation in the theory of equations. Cardano considered equations such as $x^3 + bx = c$ and $x^3 = bx + c$ to be different problems. The reason was that he allowed only positive numbers for coefficients and solutions. For equations of the third degree this represents only a small complication, but in the case of the equation of fourth degree we have seven different kinds of equations, and in the case of quintic equations fifteen. Therefore it is natural to try to reduce this complexity. It was Michael Stifel, whom we already mentioned in connection with the introduction of the symbol for the square root, who first saw how this might be accomplished. In his book *Arithmetica integra* (1544) Stifel introduced rules for the arithmetic of negative numbers, which he interpreted as numbers smaller than zero. That is a natural extension of the number concept, as negative numbers begin to play an important role as values of the auxiliary variables. Nevertheless, Stifel went further and started to use negative numbers also as coefficients of equations. This enabled him to unite all fifteen kinds of quintic equations, which formerly had to be treated separately, into one general ***polynomial form***: $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$. Thus a polynomial as a mathematical object is first found in Stifel's work. He used a simple symbolism without symbols for coefficients. Nevertheless, the basic idea reducing all the different cases into a single form, by allowing the coefficients to be negative, was decisive. A polynomial unites different *formulas* into one universal *form*.

When we start to understand the algebraic expressions as more-or-less independent formal objects, it becomes possible to accept the square roots of negative numbers simply as special kind of expressions. Even though we do not know what such expressions represent, we know how to calculate with them. This understanding is implicit in the book *Algebra*, written by Rafaelo Bombelli in 1572. Bombelli introduced rules for the addition, subtraction, multiplication and division of these new expressions, and did not ask what they stood for. This view can be found also in Leonard Euler's book *Vollständige Anleitung zur Algebra* from 1770, where imaginary numbers are called *numeri impossibiles*, because they are not smaller than zero, not equal to zero, and not greater than zero. Euler writes: „*They force themselves on our mind, they exist in our imagination and we have sufficient notions of them, because we know that* $\sqrt{-4}$ *means a number, which when multiplied by itself gives -4*". Thus even though in reality there is no quantity whose square is negative, we have a clear understanding of the meaning of the symbol $\sqrt{-4}$.

The transition from formulas to forms is also important for another reason. If we consider formulas as the basic objects of algebra, one central aspect remains hidden. A polynomial has many roots, thus in general there is not just one number satisfying the conditions of an algebraic problem. In the first stage, the stage of rules, this was ignored, because in reality, in the normal case, the number of things we are looking for is unequivocally determined. Mathematicians therefore simply ignored the existence of other roots of an equation, and as the solution of the problem they accepted the root that made sense given the context of the problem. In most cases they were even not aware that they are overlooking some solutions, because in most cases the other solutions were negative, and thus from the realistic point of view, unacceptable. In connection with the stage of formulas the situation was somewhat better. For the auxiliary equations it was necessary to take the negative solutions into account as well, because it can happen, that a negative solution of the auxiliary equation corresponds to a „true" (i.e. positive) solution of the original problem. Nevertheless, as a solution of the whole problem mathematicians still accepted only a positive number, one that gave the „number of things". Only when the bonds tying the language to reality became looser did they accept that equations generally have more roots. Thus the transition from algebraic formulas to algebraic forms was crucial for the understanding of the relation between the degree of an equation and the number of its roots.

We expect a ***formula*** to tell us the result. A formula expresses a number we want to know, it represents the answer to the question we are asking. A ***form***, on the other hand, is a function, giving different results for different inputs It might not be easy to imagine that a given problem has more than one answer, because if we are asking something about reality, we expect that the answer is uniquely determined. Yet when we understand the equation describing the problem as a polynomial form, it becomes understandable that the form can produce the same value (usually zero) for more than one value of its argument. Thus the transition from formulas to forms makes it easier to accept that an equation can have more than one solution. When the equation is understood as a form the relation between the roots and the coefficients can be disclosed, as was done independently by Albert Girard (1595-1632) and René Descartes.

In this way the language of algebra becomes a means for grasping the unity behind the particular formulas and quantities. This unity opens up a new view of equations. Instead of searching for a formula that would give us the value of the unknown, we face the task of finding ***all*** the numbers that satisfy the given form. In other words, we are searching for numbers which we can use to split the form into a product of linear factors. Consider, for example, the form

$$x^3 - 8x^2 + x + 42 \;=\; (x - 7).(x - 3).(x + 2),$$

which shows that 7, 3 and −2 are the roots of the polynomial $x^3 - 8x^2 + x + 42$. To solve the equation $x^3 - 8x^2 + x + 42 = 0$ now means to find all its roots. When we have found the roots, we are able to split the form $x^3 - 8x^2 + x + 42$ into linear factors $(x - 7)$, $(x - 3)$ and $(x + 2)$. This factorization shows that no other root can exist (for any number different from 7, 3 and −2 each factor gives a nonzero value and so their product is nonzero). Thus the splitting of the form into linear factors gives a complete answer to the problem of solving an equation. To solve an equation means to ***split a form into its linear factors***.

5. History of algebra and structure building

In history of algebra the tree layers of structure−algebra of rules, algebra of formulas, and algebra of forms− are separated by rather long periods of time. It is possible that many of the difficulties students encounter in understanding algebra are caused by the fact that these three layers are introduced quickly one after the other. Therefore one layer is not stable enough to be able to support the building of the next layer, and so the connections between them are only vague.

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APPROACHING THE DISTRIBUTIVE LAW WITH YOUNG PUPILS

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Abstract:*This paper contributes to the research strand concerning early algebra and focuses on the distributive law. It reports on a study involving pupils aged 8 to 10, engaging in the solution of problem situations, purposefully designed and presented through concrete objects, drawings, oral or written descriptions. The study focuses on ways in which* perception *leads to different mental images that influence the choice of either the (a+b)×c or the (a)×c)+(b)×c)* representation*. Our hypothesis is that understanding these dynamics is a fundamental step for the construction of a meaningful approach to properties based on suitable activities, organised so as to favour an explicit statement of proposed solutions and a collective comparison of arithmetic expressions that codify solution processes.*

Keywords: perception, mental models, language representation of processes, distributive law.

1. Introduction

This work is part of the ArAl project, which was designed to revisit arithmetic teaching in a pre-algebraic perspective (Malara & Navarra, 2001, 2003a, 2003b) and concerns a fragment of a teaching path centred on problem solving activities finalised to construct in pupils an experiential basis for an objectification of the distributive law, through collective discussions for sharing and reflection[1]. The distributive law, together with associative and commutative laws, plays a key role on both the arithmetical (mental calculations, algorithms, rule of signs, …) and algebraic side (transformation of expressions, recognition of equivalence relationships, formal identities, …) and more generally in the production of thinking via algebraic language. In usual teaching practice however, these properties are taken for granted, almost assumed as tacit axioms, or worse, they are assigned to be learned by heart from the textbook. Pupils are thus led in the position not to understand the sense of these properties, to perceive a rupture between the experiential and the theoretical, and not to recognise their value on the operative level. The tacit spreading of this phenomenon is documented by studies concerning teachers (Tirosh *et al.*, 1991) and by studies focusing on a conscious learning of arithmetical properties and of the distributive law in particular (Mok 1996, Vermeulen *et al.* 1996). In our project this property enters the game in many situations and it is exactly due to this pervasiveness that we deemed important to design a path aimed at its objectification through problem situations that highlight its genesis. Our first results highlight the influence

[1] The theoretical frame of the work is essentially the one of the project and it is sketched in the quoted papers. The English version of the project is in <**www.matematica.unimo.it/0attività/Formazione/ArAl**>.

of perception on the construction of mental images, useful for conceptualising the property, and the effectiveness of processes of sharing.

2. The situation

The class[2] which is object of the present analysis, is beginning a path which will lead to the conceptual embryo of the distributive law.

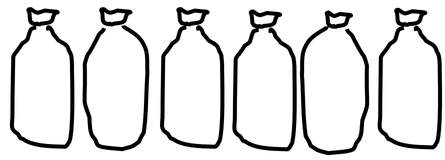
The objective is to construct premises for subsequent developments, finalised to the appropriation of the property as a *mathematical object.* The activity develops through three problem situations that favour the development of dynamics that can be summarised in three phases: 1) from confusion to the first arithmetical representations; 2) from the first perceptions to the two constitutive representations of the property; 3) reflection on the two representations and appropriation of the mutual equality of the expression values. The three situations are meant to favour the transparency of the transition from *perception* of the situation to *translation* into mathematical language. It is thus necessary to (a) *educate* pupils' perception, i.e. lead them to become aware of the existence of *diverse* ways to perceive a situation, among which some may be more *productive* from a mathematical point of view; (b) make pupils understand that it is possible – through collective sharing – to understand the meaning of translations and conceptualise their mutual equivalence beyond the process each of them identifies. Very often teachers themselves must be educated analogously.

We present here a teaching sequence, overall lasting about three hours (distributed in three sessions) to be considered as an example of the evolution of thinking in both *individual* and *collective* forms. The most meaningful parts of the diary are described in detail, whereas other parts, meaningful for their overall sense, are synthesised. As the reader will notice, the initial interventions by pupils denote a certain confusion about the assigned task and an apparent regression with respect to competencies that were acquired the previous year in the solution of problems that had a similar structure. These are consequences of the assignment, repeatedly asking *not* to *solve* the problem finding a *result*, but to *explore* one's own *modus operandi*. Confusion is thus due to an atypical didactical contract: pupils are asked to work at metacognitive level and this request, although having strong educational value, is harder to be managed by both students and teacher. One of the main features of the ArAl project is to favour reflection on processes: to obtain this, it promotes activities that stimulate metacognitive and metalinguistic competencies and construct sensitivity towards these aspects in teachers.

[2] It is a grade 4 class from Birbano (Belluno, Italy) at the beginning of the year school (2002-03). The activity was planned within a yearly cycle of meetings in which the teacher researcher Giancarlo Navarra and Cosetta Vedana, class teacher for the mathematical-scientific area were simultaneously present.

3. Phase 1: from confusion to the first arithmetical representations

(i) Problem presentation and assignment

The teacher puts 6 bags (made of a non-transparent fabric) on a desk and explains that each of them contains 7 triangles and 12 squares.	

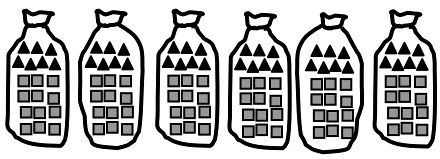
The task is *to count how many objects are in the bags totally*. It is strongly underlined that the important thing is *not the number* of objects, but rather the *reasoning process followed* to find it. Pupils know the quantities referred to objects but cannot see them, therefore they are forced to construct *mental models*. In order to do this, they must initially *focus on their perception* of the imagined situation, in an intertwining of *unstable* perceptions and *floating* calculation attempts.
The task is complex: in fact the pupil is asked not to *count*, but to *look at himself/herself in the counting act*. He/she must face a metacognitive task: reflecting on his own actions.

3.1. From confusion …

(ii) (Class[3]) The first difficulty is a psychological one: pupils are anxious and do not understand the task.

(iii) (C) The next step occurs at cognitive level: being uncertain about the task, pupils go for the most familiar interpretation and mentally count the content of a bag: there are 19 items[4]. These are in a single bag, but they show to be thinking that anyway one step is completed, because what seems to be important for them is the *number*, i.e. the 'result'.

(iv) (TR) A *visual* aid is given: pupils are invited to open the bags.



(v) (C) Still *confusion*: seeing the objects does not seem to offer significant help. While searching for an interpretation of the task, pupils start manipulating the items: they group them by colour, by shape, others leave them shuffled. However, this manipulation does not provide particular hints.

(vi) (TR) *The task is reformulated and a discussion is solicited*: "I did not ask you to tell me a number …do you remember? I asked you to count mentally the pieces and then try to explain how you proceeded for counting. Look inside yourselves, as in a movie. What did you think? Where did you start from?"

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[3] From now onwards C will stand for 'class' and TR for 'teacher researcher'.

[4] Pupils are still not seeing the items, but need to imagine them, hence 'counting' is done on a virtual context, without the reassuring feedback given by a physical contact with objects. But, as we will see later, pupils will keep interpreting the request to 'count' in a strict sense, an operative one, instead of management of a complex situation in which 'counting' may become a sort of umbrella, under which several strategies for calculation can be developed.

(vii) (C) The new discussion is still confused, but more choral and animated, with weak metacognitive features: some pupils say that they count pieces *one by one*[5], others *using the times 2 - table*, others using the *times 5 – table* (discussion about the strategies highlights that they count pieces two or five at a time, to go faster), others by *groups of colours*.

(viii) (TR) The *task is formulated again*, and pupils are invited to give less generic explanations, *taking into account the information given by the problem*: "Look carefully at what is in front of you: there are *six* bags; each bag has the same content, made of triangles and squares, and there are *seven* triangles and *twelve* squares. Your brains are working with *these* numbers".

(ix) (C) The activity evolves at *metacognitive* level: pupils, working in small groups, manipulate blocks meditating on the moves, with slow shifts accompanied by reflection; in a Gestaltian sense, pupils are *restructuring their field*, searching for meaningful perceptions. Calculation processes start shaping up in a complete and communicable way. Embryos of processes are proposed: for instance, pupils of a group say that in order to find the total number they "did 19 times 6".

Steps i – ix:
The initial situation (i) in which triangles and squares are not visible makes pupils uncomfortable (ii) and is sorted out by means of a calculation *(iii) but it forces pupils to construct mental images of the situation.* Seeing *physically the objects (iv) does not help in the beginning (v) because possibly the real problem does not lie in vision per se, but in the* organisation *of the vision itself. The repeated invitation to* look inside oneself *(vi-viii) leads to an increasing development of* metacognitive *activity and, consequently, to the elaboration of more organised attempts to 'see' the situation with the eyes of mind. Hence a virtuous circle is enacted (ix) between an increasingly 'guided'* perception *and a growing clarity in the interior visualisation of* mental processes *and in their* verbal description. *The situation is mature for Brioshi's*[6] *entry.*

3.2. … to the first arithmetical representations

(x) *Mathematical language* enters the scene: it is time to verify if and how field restructuring – and hence the game of back-and-forths between perception and development of mental models – has produced images that can be represented through mathematical language. *Brioshi* is called in: (TR) "What message could you send him to explain how you *managed to count triangles and squares inside a bag* and then *to find* the total number of *triangles and squares*?"

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5 As underlined in previous note, 'counting' still emerges as a litany.

6 Brioshi is an imaginary Japanese pupil (variably aged according to the age of his interlocutors) and is a powerful support within the ArAl project (the first Unit is completely dedicated to him). He was introduced to make pupils aged between 7 and 14 approach formal coding and a difficult related concept: the need to respect rules in the use of language, need which is even stronger when engaging with a formalised language, because of the extreme synthetic nature of the symbols used in them. Brioshi is able to communicate only through a correct use of mathematical language and enjoys exchanging problems and solutions with foreign classes, through a wide range of instruments, such as messages written on paper sheets or more sophisticated exchanges through the Internet.

(xi) (C) Part of the pupils formulate (individually) the following proposals, transcribed on the blackboard and then discussed.

(a)	**9 + 7 + 3 = 19**
(b)	**19 × 6**
(c)	**(5 + 5 + 5 + 5 + 5)**[7]
(d)	**5 × 3 + 4 = 19**
(e)	**5 × 4 - 1**
(f)	**2 × 4 = 8 + 11 = 19**

Sentences highlight a *short circuit* with the task. Except for (b), the others express a conviction that different ways of expressing the content of a bag, that is 19, must be listed. This misunderstanding leads to substantially unreasonable expressions, often impenetrable, because the authors cannot justify the reasons underlying their representation.[8]

(xii) (TR) *Description of the situation*: inviting pupils to use a representation in mathematical language was premature and natural language becomes again the mediator – with a fundamental role, given the age of pupils – through which pupils are asked to *describe* the *concrete* situation as it is.

(C) At the end of the discussion, the class comes to a collective formulation: "There are six bags, all on a desk: there are 7 triangles in each bag and 12 squares in each bag, we must represent and find how many they are altogether".

(xiii) A proposal of sending a new message to Brioshi is made, in order to take into account what has been said.

(xiv) The class formulates different proposals, showing an evolution with respect to the previous ones:

(g)	**7 + 12 = 19**
(h)	**7 × 6 + 12 × 6**
(i)	**72 + 42**
(j)	**19 + 19 + 19 + 19 + 19 + 19**

Through discussion pupils focus on (h), (i) and (j) but they see them as *different* things. They do not grasp the underlying mental models.

(xv) (TR) Pupils are asked to re-*describe* the situation in written natural language.

(xvi) (C) Some descriptions are still generic, for instance: "there are 6 bags and 2 different shapes", but two families of descriptions emerge that mark the beginning of a *turning point*: (a) "6 groups of squares and 6 groups of triangles"; (b) "6 bags, in each bag there are 7 triangles e 12 squares".

(xvii) (TR) Pupils are asked to write other sentences for Brioshi individually; two groups of sentences come out, referring to the two models:

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7 The proposal comes by a pupil with difficulties.

8 It often happens that when the task is not clearly understood, pupils that express a higher self-confidence are the least aware whereas more prudent pupils show to have a stronger critical capacity and prefer to 'stay at the window'.

(A) a x c + b x c	(k)	7 × 6	6 × 12	72 + 42
	(l)	72 + 42		
	(m)	6 × 12 + 6 × 7		
(B) (a + b) x c	(n)	19 × 6		
	(o)	12 + 7 × 6		
	(p)	(12 + 7) × 6		
	(q)	7 + 12 × 6		

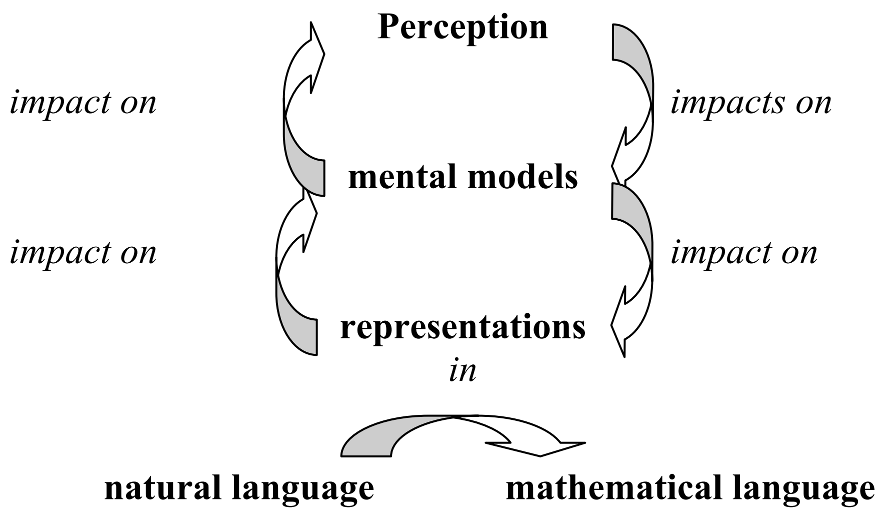
(xviii)(TR) Pupils are asked to comment on the formulations.

(xix) (C) At the end of discussion these conclusions are reached:

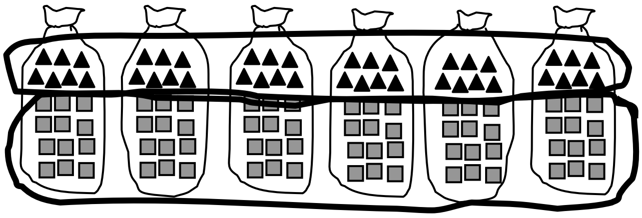
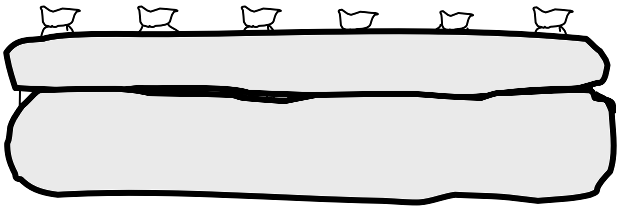
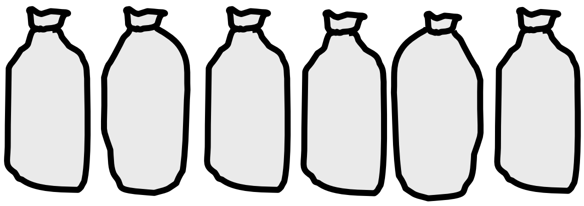
Group (A) conclusions'*… first they find the whole lot of triangles and then the whole lot of squares';* group (B) conclusions say that '*… they calculate the number of squares and triangles altogether'.*

Steps x – xix

Brioshi's entry (x) starts up an activity of representation in mathematical language; after a start influenced by a possible misunderstanding on the task (xi) the recourse to formalised and natural language alternatively (xii-xiv) produces increasingly meaningful results. A system of relationships is outlined that can be visualised through the following model:



Pupils' increasing capability in moving inside the relationships illustrated in the model leads to the production of sentences (i), (j) and (k) in (xiv), the transparency of which makes possible to trace back the organisation of perceptions that generated them. Pupils are the protagonists of this reconstruction, through which the activity is read at a metacognitive level.

(i) 7 × 6 + 12 × 6 The relationship between representation and the mental model elaborating perception is *transparent*.	
(j) 72 + 42 The representation (R) reflects the previous mental model (M) but it is *opaque*. The relationship between (M) and (R) is weak. Control at metacognitive level is scarce.	
(k) 19 + 19 + 19 + 19 + 19 + 19 The representation refers to a perceptive act, and therefore to a mental model, that although different is still *opaque*.	

Through further intertwining of natural language (xv-xvi), mathematical language (xvii) and natural language again (xix), pupils reach conclusions that introduce effectively an embryo of the distributive law (xix).

4. Improving the two representations

(xx) (TR) A new problem is proposed (a week later):

> Granny prepared for Santa Lucia 8 bags of sweets for her nephews.
> In each bag she put 5 chocolates and 14 candies.
> How many sweets did granny buy?[9]

(xxi) (C) Pupils solve it with no questions about clarification. Two types of solutions are provided and they can be ascribed to both representations:

(A) a x c + b x c (5 pupils)
- **(r) 5 × 8 = 40**
- **(s) 14 × 8 = 112**
- **(t) 112 + 40 = 152**

(B) (a + b) x c (6 pupils)
- **(u) 14 + 5 = 19**
- **(v) 19 × 8 = 152**

Models (A) e (B) are nearly equally distributed; proposed calculations are all carried out separately until the result is obtained.

(xxii) (C) During the discussion two pupils provide decisive contributions:

(A_1) Denise wrote in a rough copy:

$$5 \times 8 + 14 \times 8 =$$

[9] The question 'How many sweets …' although focusing on the outcome and not on making the process explicit is nevertheless clear to pupils due to the established contract.

but she did not know how to continue and preferred to go back to single operations. She explains she recognised the same problem she tackled previously.

(B_1) Giada realises that Denise's procedure is the 'translation' of the first solution type and tries to translate the second type solution, writing:

$$14 + 5 + 19 \times 8$$

but she realises that it is not good. Collective discussion helps her to modify it:

$$(14 + 5) \times 8.$$

Steps xx – xxii

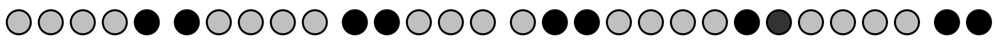
Presentation of a new problem (xx) raises two types of representations by separate steps, which can be reduced to those of distributive law (xxi); during discussion representations in a line appear (xxii). The former representations are blocking, whereas the latter constitute a fertile *ground.*

Leading pupils to representations in a line *seems to be a necessary condition (although not a sufficient one) to construct a mental attitude that may favour the transition to an embryonic view of the property. As we said earlier, this condition is subordinated to an* education to perception *of elements of the problem situation. At a first level, most pupils are attracted by aesthetic, formal and expressive aspects that distract them from the logico-mathematical aspects. Denise and Giada are probably two among the few students that show a natural inclination for selective analysis. Generally,* education *plays a determinant role: this means leading the class, through sharing, to make perceptions and reasoning explicit, so that differences may become productive for a collective construction of shared knowledge.*

5. Reflecting on the two arithmetical representations

(xxiii)(TR) A week later, a third problem situation is proposed:

A giant cardboard necklace made of alternating four grey beads and two black beads is shown:



The task is the usual one: to explain in either natural language or mathematical language (or both) the *way* in which one can find how many beads compose the necklace.

Again pupils must try to *describe* what they are *thinking*.

(xxiv)(C) Proposals are compared and commented upon:

(Giulia) I count how many the beads are: $2 \times 5 + 4 \times 5$
but 'how' did you count? [note written by the teacher]: I counted this way: the beads are thirty and to make this result I counted them with multiplication.[10]

[10] For many young pupils the verb 'to count' has a similar meaning to the verb 'to calculate'. Perhaps to Giulia the two verbs express the same action, the same content, and this action and content can be expressed only in mathematical language, or rather: to her the latter is the most 'spontaneous' way to find the number of beads. The activity is carried out at cognitive and not metacognitive level.

(Lorena) I calculate how many the black beads and the grey beads are: $2 \times 5 + 4 \times 5$[11]
(Claudia) Every two black beads there are four grey beads[12]: $2 \times 5 + 4 \times 5$
(Giada) Two black ones and then four grey ones[13]: $2 \times 5 + 4 \times 5$
(Alberto) $(2 \times 5) + (4 \times 5) = 10 + 20 = 30$ I did 2 the number of black beads and I multiplied it by 5, same thing for four.

The class realises that everybody used formulations of a single type (A)[14].

(xxv) (TR) Pupils are invited to express the situation with the *other* mathematical formulation. Alberto proposes, raising general satisfaction in the class:

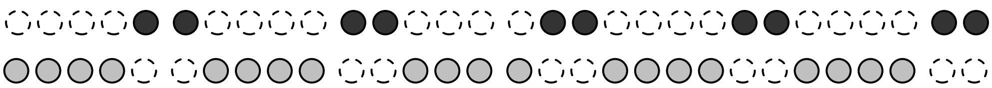
$$(2 + 4) \times 5$$

(xxvi) (IR) The class is asked to explain how the necklace was "viewed" by those who wrote $4 \times 5 + 2 \times 5$ and Alberto, who wrote $(2 + 4) \times 5$.

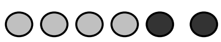
(xxvii) (C) Giulia: "We count how many the black beads are and then the grey ones and put them together"[15]. Giada: "Alberto adds the four grey beads to the two black beads and repeats them 5 times".

(xxviii) (IR) A 'mental experiment' is proposed to the class: "Imagine a completely dark place where you can switch on a spotlight to illuminate the things you want to highlight every time. In this dark place there is your necklace: draw a sketch showing, as under the spotlight, the necklace seen in the first case, i.e. by the class, and then another sketch showing the necklace in the second way, i.e. seen by Alberto."

(xxix) After some uncertainty, pupils highlight the 'two moments' in which the necklace is perceived in the first case. They draw two sequences of beads in which they highlight separately the beads of different colours leaving the others white:



The necklace 'seen' by Alberto needs 'one single moment' in which the spotlight highlights the repeated module.

Stencils and friezes are recalled: pupils agree that in the first case two stencils are needed (), and in the second case one stencil is enough (). A pupil says that it is more convenient and recalls the already encountered economy principle.

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[11] Lorena suggests that, once explained what *she does*, numbers and mathematical signs express *how you calculate*.

[12] This is the description of what she sees in the necklace: is it also the "description" of the way in which she gets to the solution?

[13] See previous note.

[14] The pupils' sentences reveal the dominance in the perception of the black colour, it leads the pupils to overcome the sequential order of the beads.

[15] Again Giulia uses the verb "to count" as condensing actions, operations that express "the way in which" in a compact, condensed way.

(xxx) The need for a mathematical expression comes back to make Brioshi understand that the two ways of ‘seeing’ the necklace are *equivalent*. Pupils are quick in proposing the following expression:

$$4 \times 5 + 2 \times 5 = (2 + 4) \times 5$$

Steps xxiii – xxx

A third problem (xxiii) leads to representations referring to the only expression (A) $a \times c + b \times c$ *(xxiv), although the formulation of the text seemed to induce (B):* $(a+b) \times c$*. The ‘dominant’ perception confirms what emerges from other activities of the ArAl Project, concerning the search for regularities. In front of a sequence (frieze, necklace, etc.) characterised by alternating groups of elements, for instance two, pupils identify* alternation *more regularly than* repetition *of a module made of both groups. The hypothesis we formulate is that perceiving* independent *elements is more spontaneous than perceiving* relationships *between elements*[16]*. Perception of the alternation hinders the identification of the* structure *of the sequence and inhibits representation (B). A* field restructuring*, in Gestaltian terms, is necessary.*

The teacher’s invitation leads to the emergence of (B) (xxvi) and to a verbal description of the mental models underlying (A) and (B) (xxvii-xxviii). An ‘experiment’ is proposed to favour a re-reading of the context (xxix): this leads the class to elaborate on visualisations that make the two different perceptions transparent *(xxx) and to an intuition of the equality of the two representations.*

Conclusions

We now simply give a short indication about the prosecution of the didactical path. The key point is focussing pupils’ attention on the comparison of the arithmetical writings arising from the solutions of faced problems, in order to lead them to grasp the general validity of the equality $(a+b) \times c = (a \times c)+(b \times c)$. The main steps of this part of the path are: a) problem situations with iconic support differing for both context and numerical values, in order to favour the two different perceptions of the field; b) problem situations similar to the previous ones, without iconic support that differ for both context and numerical values; c) problem situations proposed in two partially different versions, in order to strengthen the sense of the two representations; d) comparison among problem situations and the related expressions representing their solutions, in order to favour the understanding of the independence of equalities from numerical values and types of data; f) framing of the various equalities in a scheme and conceptualisation of the property. The detailed analysis of these steps of the path and the reflections about the ways in which the pupils conceptualize the property will be the topic of another paper.

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[16] Another hypothesis is that, since ‘seeing’ is a procedural activity, the diversity of colours breaks the perception of the unity of a module, highlighting two subsequences, and this would induce a *distributed* vision (*A*). The other one (B) is more evolved because it concerns a vision that goes beyond colours and captures the *unitary structure of the bicolour module*. The two hypotheses are being compared and analysed in depth.

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THE DEVELOPMENT OF INFORMAL PROPORTIONAL THINKING IN PRIMARY SCHOOL

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Abstract:*The purpose of this study was two-fold. First to describe students' informal strategies for the solution of proportional problems and second to describe students' conceptual understanding of proportion. Kaput and West's (1994) informal levels of proportional reasoning and Sfard's (1991) levels of conceptual development provided the framework for this description. A test and semi-structure interviews were used to gather data from 112 ten and eleven year old students. The data suggests that students use a variety of informal strategies to solve proportional problems. The results also reveal that students who have not received any formal teaching on proportion, may exhibit characteristics of internalization, condensation and even reification of the proportion concept.*

Keywords: informal levels of proportional reasoning, reification theory, conceptual development.

INTRODUCTION

Proportional reasoning is at the heart of middle grades mathematics. Due to the importance of proportional reasoning, numerous studies (Christou & Philippou, 2002; Tourniaire & Pulos, 1995; Kaput & West, 1994) have investigated students' strategies in their attempt to solve missing-value proportional problems. Missailidou and Williams (2003) have classified missing value proportional problems according to their level of difficulty, while Kaput and West (1994) proposed three broad hierarchical categories of students' informal but competent strategies for the solution of proportional problems.

However, so far no major attempt has been made to investigate students' conceptual understanding of the proportion concept before formal instruction. In this study we try to investigate the development of students' conceptual understanding of proportion before formal instruction. We have used Kaput and West's (1994) informal levels of proportional reasoning and Sfards' (1991) levels of concept development as a guiding framework for explaining the structure of students' informal proportional thinking. The underline assumption is that the structure (Hejny, 2003) of students' informal proportional reasoning is an important factor to their subsequent development.

Kaput and West (1994) proposed three broad types of competent but informal proportional reasoning levels that consist of patterns of reasoning that support the solution of missing-value problems without reliance on the syntactic manipulation of formal algebraic equations. The first level of proportional reasoning referred to as the coordinated build-up/build down approach includes steps like distinguishing between the two referents A and B to be quantified in the problem situation (ie., A and B are pencils and cents) and constructing a semantic correspondence relation between the classes of referents A and B at a gross level (i.e., 4 pencils corresponds to 40 cents). It also includes the action of constructing a correspondence relation between respective units at the group level and distinguishing between the third given quantity and the fourth unknown quantity by linking each to its respective referent type A or B (i.e. If 4 pencils corresponds to 40 cents then 5, 6, 7 … pencils to 50, 60, 70,…cents). The computation students follow at this first level, is the increment or decrement of both quantities until the third given quantity is reached and then identify its corresponding element of the other quantity as the problem's solution. The second level of informal proportional reasoning called abbreviated build up/build down approach differs from the first level in the computation used. Students in the second level divide the total given quantity by the quantity per unit to obtain the number of units and then they multiply the number of units by the corresponding quantity per unit to determine the total unknown quantity (i.e. If 4 pencils corresponds to 40 cents then 7 pencils will correspond to 40 : 4 =10 and 7X10 = 70 cents). Finally, the third level includes the unit factor approach in which students divide the unit size of the unknown quantity by the unit size of the known quantity to determine the unit factor. Then they multiply the unit factor and given total quantity to determine the total amount of the unknown quantity (i.e. 15 cans paint 18 chairs, then 25 cans will paint 18/15=6/5 6/5X25 = 30)

Sfard's theory of reification (1991) describes three levels of mathematical conceptual development. She argues that when the learner is at the stage of interiorization s/he gets acquainted with the processes which will eventually give rise to a new concept. She argues that these processes are operations performed on lower-level mathematical objects and that the learner becomes gradually skilled at performing these processes.

At the second stage of concept development which is called condensation, the learner becomes more and more capable of thinking about a given process as a whole without feeling an urge to go into details. At this stage a new concept is formally born. Condensation should be regarded as the stage where processes defining the concept become more concise for the learner and the learner becomes increasingly capable of dealing with alternate forms of the concept (Goodson-Espy, 1998).

The third stage which is called reification entails that a learner is able to conceive of a concept as a 'fully-fledged object' (Sfard, 1991, p.19). She explains that various representations of the concept are unified in the students' reified construct and the construct is no longer dependent upon a process. The student at this stage is able to attribute meaning and significance to the construct by understanding the conceptual category in which it belongs. The reified concept is now ready to be used as an input in higher-order processes that can lead to even more powerful constructs.

Interrelationship of Sfard's theory and Kaput and West's levels of informal proportional reasoning

Sfard's theory aims to provide a way of describing the nature of mathematical conceptions that a student holds. A significant question involving this theory is what prompts the transition from one stage to the next in the context of proportional thinking. In this paper it is suggested that Kaput and West's levels of informal proportional reasoning may be used to illustrate how such transitions may take place. Particularly, in order to demonstrate how the subject's conceptual structures could be described, we hypothesise that the levels of informal proportional reasoning defined by Kaput and West (1994) can be connected to the stages of the reification theory. In this section we will discuss the way in which we believe that the two theories are linked. This interrelationship of the two theories was examined in this study.

In the interiorization stage Sfard (1991) argues that the learner gets acquainted with the processes and that these processes are operations performed on lower-level mathematical objects. Moreover the learner becomes gradually skilled at performing these processes. We hypothesise that Kaput and West's (1994) first level of informal proportional reasoning (i.e. the coordinated build up/build down approach) characterizes students at the level of interiorization since students attempt to solve these problems mainly with the use of repeated additions or subtractions.

Sfard (1991) describes condensation stage as the stage in which a person becomes more capable of thinking about a given process as a whole without feeling an urge to go into details. In the present study students' problem solving activities were examined for connections to the process that the students had used to solve the tasks. These connections were classified based on whether the students only reused the first method described by Kaput and West (1994) or whether the students moved to the second level of informal proportional reasoning i.e. used a less detail process to solve the proportional problem. In addition, another characteristic of students' solutions indicating that they belong to the condensation stage is the flexibility and the variety in the strategies that they use to solve different proportional problems.

In the reification stage students are able to operate on a concept without depending upon a process. The student at this stage is able to attribute meaning and significance to the construct by understanding the conceptual category to which it belongs. In the present study the problem-solving activities of students were examined for evidence of the third level of informal proportional thinking described by Kaput and West (1994). This level includes the unit factor approach as described above. In addition to this, a student can be classified in the reification stage if s/he can recognize different representations of the same proportional situation and if he could categorize a problem as a proportional one without carrying out the computations.

In order to capture the structure of students' proportion concept this study investigated two questions:

- How can students' solutions to proportional problems be classified in the levels of informal proportional reasoning identified by Kaput and West, (1994)?

- What are the characteristics of the students at each one of the three conceptual levels proposed by Sfard (1991) in regard to the concept of proportion?
- Can Kaput's and West's (1994) levels of informal proportional reasoning be used to illustrate the transition from interiorization, to condensation and finally to reification (Sfard, 1991)

METHOD

The study was both quantitative and qualitative in nature. More specifically, a test consisting of proportional problems, was administrated to 112 students in grades 5 and 6 (10 and 11 year olds). In addition, interviews were conducted with 3 students which were identified based on their responses in the test. It was hypothesized that each student belonged at each one of the three conceptual levels determined by Sfard (1991). The data was collected before students received any formal instruction on solving proportional problems. All of them had been taught addition, subtraction, multiplication, division both with whole numbers and fractions.

The design of the test was crucial to the identification of students' conceptual understanding of proportion. The test consisted of three parts. The first part included tasks, which were used in recent studies (Kaput & West, 1994; Misailidou & Williams, 2003). The main purpose of part A was to investigate if students were able to use different strategies for the solution of the problems and to classify students' strategies in the three levels proposed by Kaput and West (1994). The five problems used in the test (see Table 1) could be solved using various informal strategies described in the literature (Christou & Philippou, 2002)

1. John has canaries and parrots. For every 4 canaries he has 3 parrots. If all the canaries that John has are 28 how many parrots does he have?
2. To make Italian dressing you need 3 parts of vinegar for 8 parts of oil. How much vinegar do you need for 96 ml of oil?
3. At a fruit stand, 3 apples cost 90 cents. You want to buy 7 apples. How much do you have to pay?
4. Mary bought 6 books and paid £4. Elena bought 24 books. How much did Elena pay?
5. George used 15 cans of paint to paint 18 chairs. How many chairs will George paint using 25 cans of paint?

Table 1: Test, Part A-Proportional problems.

Solutions to the problems

If the child solved problem one with the build up/down strategy described by Kaput and West (1994) this was characterized as a process at the level of interiorization. The second problem involves larger numbers and so the application of the build/up down approach is time consuming. Thus students could solve it by applying a different strategy to the one used in the first problem something that will lead them to a higher level of informal proportional reasoning. The third problem could be solved by first finding the price of one quantity and then multiply the total amount of quantities with the price of one (*unit-rate approach*). The most suitable approach for the fourth problem was the *factor of change approach* (4 times more). In this study, the unit-rate approach and the factor of change approach, were classified at the second level of informal proportional reasoning because even though they are multiplicative in nature they do not involve the sophistication of thought described by Kaput and West (1994) as necessary for the third level of informal proportional reasoning (Nabors, 2003). These two problems were also included in the test in order to examine students' flexibility in using a variety of strategies for the solution of the problems. Sfard (1991) argues that learners belong to the condensation stage when they flexibly use different strategies for the solution of a given problem.

The fifth problem was the most difficult one. This was documented in Misailidou and Williams's work (2003). In order to solve this problem students should be able to use efficiently the unit factor approach as described by Kaput and West (1994) because the problem involves the continuation of context in which whole number quotients are not required (Nabors, 2003).

Students were categorized in the levels of informal proportional reasoning based on the strategies used to solve the problems in part A. The categorization of students to these three levels was also allowing us to clearly state the strategies that students apply at the internalization, condensation and reification levels. However, we still needed to investigate whether students were able to use different kinds of representations and whether they were able to identify the proportion concept in different problems without solving them.

Part B of the Test consisted of two problems similar to problems 1 and 4 of part A, but this time accompanied by specific diagrams. Sfard (1991) argues that a criterion of students' conceptual development in the reification level is their ability to recognize multiple representations of the same concept. Thus we wanted to investigate students' ability to solve proportional problems when presented with the use of diagrams.

Part C of the Test required students to group the problems that were similar without solving them. Four problems were presented. The 1st problem was a proportional problem while the second problem was a pseudo-analogical problem (Gagatsis, 2003). Its given quantities were a boy's age (10 years old) and his height (1,42 cm). The question was "what will his height be when his age is 20". The 3rd problem was a proportion problem and the 4th problem was not a proportion problem. Sfard (1991) argues that students who are at the reification level are able to attribute meaning and significance to the construct by understanding the conceptual category to which it belongs without depending upon a process. Another, characteristic of students' at the level of reification is their ability to use the unknown quantity in their solutions. Thus moving away from arithmetic and showing some understanding of algebra.

Each one of three parts of the test can clearly indicate "what" and "how" a student answers at proportional problems. However, students' responses in all three parts and an interview are needed in order to clearly argue at which level of conceptual development a student belongs to. Thus three students, one at each level of concept development, were identified from their responses in all three tests in order to give a clear picture of students' characteristics at each level. The students were interviewed with the use of semi-structured interviews for approximately 50 minutes. The interviews were audio-recorded. First, the students had to explain their strategies in part A of the Test. Then they had to recognize which of the problems in part A were similar to the problems in part B and justify their decision. In part C of the test, students had to explain the way in which problems could be grouped.

RESULTS

In regard to the first questions of the study, students' correct responses to the problems accompanied by a correct explanation were categorized according to Kaput and West's (1994) levels of informal proportional reasoning (Table 2).

Problems	**Wrong responses % N=112**	**Correct /no explanation % N= 11 2**	**Kaput &West -1^{st} level % N=112**	**Kaput & West 2^{nd} level % N=112**	**Kaput & West 3^{rd} level % N=112**
α1	**30,4**	**9,8**	**10,7**	**49,1**	**0**
α2	**64,3**	**1,8**	**5,4**	**28,6**	**0**
α3	**35,8**	**8**	**1,8**	**53,5**	**0,9**
α4	**51,7**	**6,3**	**5,4**	**35,7**	**0,9**
α5	**70,5**	**7,2**	**1,8**	**0**	**20,5**

Table 2:Students' correct responses classified to the three levels proposed by Kaput and West.

Most of the correct responses to the 1^{st} problem were at the 2^{nd} level of proportional reasoning (49,1%). In order to find the number of parrots students classified in this level divided the total given quantity (28 canaries) by the unit quantity (4 canaries) to obtain the number of units (7) and then they multiply the number of units by the corresponding quantity per unit (3 parrots) to determine the total unknown quantity (21 parrots) (Fig.1). 10,7% of the students applied the building up approach (1^{st} level of informal proportional reasoning). Students' responses classified at this level involved the increment or decrement of both quantities (canaries and parrots) until the third given quantity was reached and then identified its corresponding element of the other quantity as the problem's solution (Fig.2).

Figure 1: Student's strategy-2^{nd} level

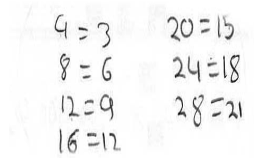

Figure 2: Student's strategy-1^{st} level

In problem 2 most of the students' correct responses were classified at the 2nd level of proportional reasoning (28,6%). Half (53,5%) of the correct responses in the 3rd problem were classified at the 2^{nd} level. In problem 4 most of students' correct responses (35,7%) were classified at the 2^{nd} level since students were using the factor of change approach (4 times more books). Some students (5,4 %) used strategies classified at the 1st level using the build up/down approach. Only one student gave an answer classified at the 3rd level of informal proportional reasoning (Fig. 3). This student determined the unit factor (2:3) and then multiplied the unit factor and the given total quantity (24 books) to determine the unknown quantity (16 pounds).

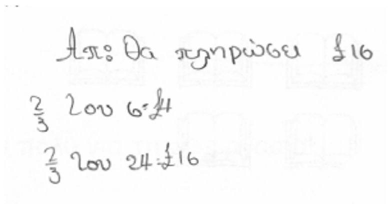

Figure 3: Problem 4 -3rd level

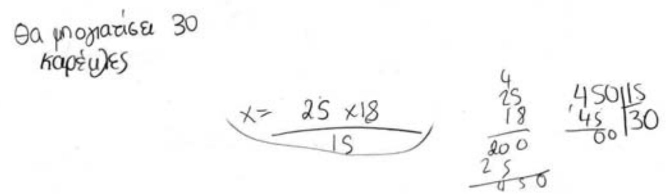

Figure 4: Problem 5-3rd level

Most of the responses in the 5th problem were classified at the 3rd level (20,5%). Students divided the unit size of the unknown quantity (18 cans of paint) by the unit size of the known quantity (15 chairs) to determine the unit factor (Fig.4). Then they multiplied the unit factor and the given total quantity (25 cans of paint) to determine the total amount.

In regard to the second question of the study, Table 3 shows the percentage of students that were able to solve correctly the whole of Part A, B and C respectively.

Part of the test	Correct responses-% N = 112
Part A	8,9
Part B	49,1
Part C	52,7

Table 3: Students' achievement in the three parts of the test.

8,9% of the students solved correctly all the problems in test A while 49,1% solved correctly all the problems with the diagrams in part B of the test. 52,7% succeeded in recognize the proportional problem amongst problems that were not proportional in part C of the test. Only 10 students out of the 112 solved the whole of part A correctly. Out of these 10 students only 8 also solved correctly the whole of Part B. And finally only 6 students solved correctly all of the three parts of the test.

Based on their responses to the test, 3 students, one at each level, internalization, condensation and reification, were chosen to be interviewed. The interviews aimed to highlight some of the conceptual characteristics of each level

Student at the Interiorization level

The student classified in the interiorization level used approaches classified in the 1st level of informal proportional reasoning for the solution of all the problems in part A. This is how she described her strategy for problem 2:

> X: *I added 8, 8, 8,…until 96 for the oil and then under each 8 I wrote 3 for the vinegar. Then I added all 3s that I wrote under the 8s.*

The girl did not show any flexibility in the strategies used. She applied the build up/down approach to all the problems. It is hypothesized that her fixation on this strategy was the main reason for her inability to solve the last problem:

> X: *15 cans of paint can paint 18 chairs. The difference is 3. I added 3 to the 25 cans of paint and I found that the 25 cans can paint 28 chairs.*

The girl could not identify the problems in Part A that had the same structure as the problems in Part B. What she did was to match problems according to similar surface characteristics of the problem, such as their story.

The girl could not recognize which problems were similar in part C and described a proportional situation. She was unable to see that the answer to the problem involving the height and the age of the boy was not logical.

Student at the Condensation level

The student classified at the condensation level used mostly strategies of the 2nd level of Kaput and West's (1994) classification.

> M: *I thought that for every 3 parts of vinegar it needs 8 parts of oil. I divided the 96 ml of oil with 8 and then I multiplied the quotient by 3 to find out how much vinegar he needed.*

Even though this girl applied the same strategy in the 1st, 2nd and 3rd problem, she changed her strategy in the 4th problem, although the 3rd and the 4th problem intentionally looked very similar. More specifically, she used the build up/build down approach, increasing both quantities until the third given quantity was reached and then identify its corresponding data of the other quantity as the problem's solution. The flexibility in her solution methods indicated that her conceptual development for proportion was at least in the condensation level. In order to verify this, we asked the girl to solve some problems in part A using a different approach and she was able to do so. Her strategies were oscillating between the two first levels of proportional reasoning described by Kaput and West (1994).

The girl faced some difficulty in recognizing the similar problems in part A and part B. She mainly focused on the superficial characteristics of the problems (same story) and not to the structural ones. The girl managed to find the two similar problems in part C but could not justify her decision. The girl was also unable to use the unknown quantity for the solution of the problems.

Student at the Reification level

The girl classified in the reification level of conceptual development was one of the six students who solved correctly all three parts of the test. This girl had a repertoire of approaches for solving the proportional problems. She could use flexibly different strategies for the solution of the problems in part A oscillating amongst the three levels described by Kaput and West (1994). More specifically, she used the abbreviated built up/down process for the solution of the 1st problem. She then shifted to the strategy described in the 1st level of informal reasoning for the solution of the 2nd problem. She used the unit-rate approach for the solution of the 3rd problem and the factor of change approach for the solution of the 4th problem. She then managed to reach the 3rd level of informal proportional reasoning in her attempt to solve the 5th problem. Her flexibility in using various strategies for the solution of the problems indicated that she had at least reached the stage of condensation.

This student was also successful in identifying the problems in part A which were similar to the problems in part B with the diagrams. In addition she could also identify the similar problems in part C and was able to justify her choice. The student was able to use the unknown quantity to solve the problems.

I: If we use X as the unknown quantity in the 1st problem how can we solve it?
P: Can we use X instead of 28?
I: Is 28 the unknown quantity?
P: No the unknown quantity is the canaries. So X= 4 . ..... =28
X= (28:4). 3. or X. 3 = 28:4.

DISCUSSION

The findings of the present study reveal that students are able to solve proportional problems even without any formal instruction. The students used various strategies for their solutions which can be classified in the three levels of informal proportional reasoning by Kaput and West (1994). In problems a1, a2, a3 and a4, the majority (approximately 87%) of the students' correct strategies were classified in the second level of proportional reasoning while the majority (92%) of students' strategies in problems 5 were classified in the third level.

This paper describes how the Kaput and West's levels of proportional thinking could be used to help make an assessment of a student's degree of concept formation or their cognitive structure (Hejny, 2003) concerning proportionality. The results showed that these informal levels could serve to pin point the strategies that students apply at the levels of interiorization, condensation and reification of the proportion concept. Especially noteworthy is the fact that students who have not received any formal teaching on the concept of proportion are not confined to the stage or interiorization but also exhibit characteristics of condensation and even of reification.

The findings also suggest that the informal levels of proportional thinking linked with some distinct characteristics defined by Sfard (recognition of different representations, flexibility in the strategies used and justification of choices according to the similarity of the problems) would better describe students' conceptual understanding of proportion. The strongest evidence of student's higher conceptual understanding (reification) of proportion is their ability to use strategies classified in levels 2 and 3 defined by Kaput and West (1994) along with their flexibility in the strategies that they used. The students' ability to recognize various representations of the same problem and their capacity to identify similar proportional problems did not provide sufficient evidence of their conceptual understanding of proportion. It can be argued that the combination of all the above characteristics i.e. flexibility with different strategies, recognition of different representations, ability to recognize proportional problems and the use of the unknown quantity in the solution of the problems best describe learners at the level of reification.

Overall, it can be said that students can be classified in the stage of interiorization when they use strategies according to the first level of Kaput and West's hierarchy. Other distinct characteristics of the interiorization stage that they were revealed in this study were students' inability to use more that one strategies for the solution of a problem, their inability to justify adequately their solution methods and their direct manipulation of the problem's data.

Students can be classified in the condensation level when they use various methods to solve the problems, mainly categorized in the second level described by Kaput and West. Another characteristic of this level it was found to be students' sufficient explanation of their solution strategies. Students' inability to find the similar problems given in different representation format and their inability to use symbols in the solution process may reveal that they did not progress out of the stage of condensation to the stage of reification.

The key characteristics of the reification level concerning proportional development may be students ability to work both using arithmetical methods and algebraic symbols and secondly students' use of the unit factor approach as described by Kaput and West in order to solve the proportion problems. Other characteristics of this stage may be students' repertoire of strategies and their ability to use the most appropriate strategy for the solution of a problem depending on its context. Students classified in this stage captured the concept of proportion. We may say that at this stage they have cognitively structured the concept of proportion (Hejny, 2003). This may give them the advantage to justify their thoughts efficiently, to recognize similar problems given with different representational formats and to use algebraic symbols for the problems' solutions.

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14 YEARS OLD PUPILS' THOUGHT PROCESSES: A CASE STUDY OF CONSTRUCTING THE TRIANGULAR INEQUALITY

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Abstract: *The aim of this case study is to examine pupils' thought processes in constructing mathematical knowledge. The study was carried out with twelve 14-years-old pupils. Triangular inequality was chosen as a topic and some materials were used to construct the knowledge. The data analysis showed that isolated wrongly constructed knowledge in pupils' minds may cause difficulties in constructing new mathematical knowledge.*

KeyWords: constructing mathematical knowledge, concept formation, geometry.

1. INTRODUCTION

The quality of learning can be improved when the learners construct their own knowledge. This does not mean that the learners must be alone during the learning process as Orton and Frobisher (1996:18) cited from Richards, 'students will not become active learners by accident but by design'. Examining how pupils construct the knowledge and develop the concepts can shape this design. In other words, this lies behind the constructivism. It is possible to give effective teaching which takes into account how a pupil learns as well as how their thinking ways work in developing knowledge.

The main aim of this study is to design tasks (activities) which aim at constructing knowledge in pupils' minds. Now a question arises: does this activity achieve its purpose? What kinds of obstacles do we face while constructing a piece of knowledge in pupils' minds by means of this activity? So, the aim of this study is focused on the construction of mathematical knowledge and thought processes during the activity.

In mathematics education, several researchers have tried to describe how mathematical knowledge is understood and formed. They tackle this process from different perspectives. One team, Dubinsky and his colleges, describe this process as encapsulation and formulate APOS (action-process-object-schema) theory. Actions are physical or mental transformations on objects. When the actions become intentional, they are characterized as processes that may be later encapsulated to form a new object. A coherent collection of these actions, processes, and objects, is identified as schema (Cottrill et al., 1996, cited in Gray et al., 1999, p.115). Sfard described this process in three steps: *interiorisation* of the process, then *condensation* as a squeezing of the sequence of operations into a whole, then *reification*- a qualitative change manifested by the ontological shift from operational thinking to structural thinking (Gray, Pitta, Tall, 2000, p.5).

Hejny (1989) develops a model of the process of construction of a piece of mathematical knowledge in an individual's mind. This model consists of six stages. These are motivation, isolated models, generalization, universal models, abstraction and abstract knowledge. These are described and discussed in Hejny's paper (2003). The cognitive process starts with pupil interests and acquisition of an initial set of experiences and develops towards the construction of a new, deeper and more abstract concept (Hejny, 2003).

In this study Hejny's model is used to analyse the data coherently. It helps to understand pupils' thought processes and their behaviour.

2. THE PURPOSE OF THE RESEARCH

The research design is based on constructivism. The aim of this research is to examine pupils' construction of mathematical knowledge and to examine pupils' thought processes when they are discovering 'triangular inequality'.

3. METHOD

In this research, using case studies was chosen as the main approach. It is not possible to collect the required data by separating from its context. It is not possible to understand pupils' thought processes outside the classroom. It needs to be evaluated within its context when the pupils are engaged in the tasks.

Participant observation is chosen as a main data collection tool. In addition to this observation, probe type questions and some developed worksheets were used in order to elicit pupils' ways of thinking in constructing the concept. The main aim of using observation is to record pupil-pupil, pupil-teacher (or researcher) and pupil-materials interactions as fully as possible. To meet all these demands, video camera and audio recorder and researcher field notes are used.

3.1. Participants

12 pupils, 4 girls and 8 boys, participated in the study. They were 14 –years- old students in a primary school which is located in a poor socio-economic area in Turkey. The pupils participated in the study on a voluntary basis. According to the mathematics teacher, the pupils are all average in mathematics. The pupils have not been taught the topic before.

3.2. Materials

In the study, 21 sticks of which lengths are between 5cm and 25cm are labeled according to their lengths and 5 worksheets are given. Additionally, a guideline explaining the aim of the work also is prepared. The following is written in the guidelines:

"You have got a very important task. The mathematical rule regarding the question 'which of the lengths of sticks could come together in order to form a triangle' is lost and it has to be found immediately. Your mission is to find this rule via utilizing the given sticks. Come on hurry up! The fate of this important rule of mathematics is in your hands!"

In the first worksheet, the pupils were requested to make triangles by using the sticks and to write down their trials. In the second worksheet they were requested to choose 5 samples from the list and to show which ones form a triangle and which ones do not, and explain why. In the third worksheet, the pupils were asked to determine

- the relation between the sum of the lengths of the first and second sides and the third side,
- the relation between the sum of the lengths of the first and third sides and the second side and then,
- the relation between the sum of the lengths of the second and third sides and the first side.

In the fourth worksheet as in third worksheet the pupils were requested to determine the relation of the absolute value of the difference of the lengths of the sides. In the last one, they were requested to reach a generalization depending upon the measurement results they acquired. The main purpose for designing the worksheets is to allow pupils to discover the rule of forming a triangle.

3.3. Procedure

The pupils who participated in the study were separated into groups. Each group consisted of four pupils. Each group was considered as a case. At the beginning of the study, the guideline and the sticks were given to the groups. Then the five worksheets were given respectively. Two researchers and the mathematics teacher took roles in helping the pupils to understand the activity and eliciting the pupils' thinking process. A video camera and audio recorder recorded the work of each group.

3.4. Analysis

Before the data was analysed, each videotape was fully transcribed into verbal data accompanied with the actions of students. In transcribing the conversations from the videotapes, the non-verbal interactions between pupil-pupil, pupil-researcher and pupil-material also were described in detail. Audiotapes were also fully transcribed and supported by the researchers.

All recorded conversations were analysed for the content of the talk and through interpretation of its meaning in its context. The results that were drawn from the study can be checked by cross-case comparison.

There are similarities among the thinking process of the pupils in discovering the relationship among the sides of a triangle. Taking these similarities into account, the researchers held a discussion on the findings.

4. FINDINGS

4.1. Pupils' thought processes while selecting the sticks

After understanding what the task is about, the pupils begin to examine the sticks in the envelope. When selections are analyzed, it can be observed that they especially choose the sticks whose lengths are close to each other. It can be stated that the pupils conclude that "if there is a little difference between the lengths of the sides, they form a triangle". When the pupils were half way through their work, it is seen that they select sticks through understanding what it would be like if different selections were made via reconsidering the case. The students try to establish which lengths of sticks do not form a triangle.

4.2. Pupils' thought processes in making a triangle

At the beginning of the task the pupils have a triangle shape in their mind. They have previous knowledge related to triangles. But, up to this task, they did not think what the conditions were to form a triangle related to its sides. They know a triangle has three sides and is a closed shape. All this knowledge is related toitheir previous experience. But, it was observed that the triangular shapes in the pupils' minds are as in Figure 1. The pupils want to place the sticks and make the triangle according to the triangular shape in their mind and this causes trouble with some length of sticks. For example, the pupils cannot make a triangle side lengths of which are 23-24-10 cm.

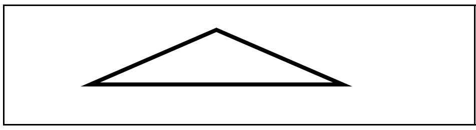

Figure 1

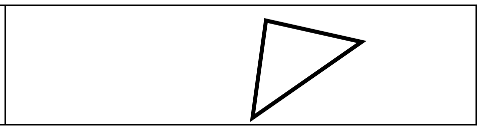

Figure 2

The pupils state that it does not form a triangle because it is not in accordance with Figure 1. Because of their opinion a triangle in Figure 2 could not be formed although they could have obtained one such as in Figure 1.

According to Hejny's (2003) model, the pupils in the second stage have isolated models related to triangles. But the pupils do not make any linkages between these models. The pupils have experience with triangles and these experiences make triangle image as in Figure 1. They did not make a link between the knowledge related to a triangle (has three sides and a closed shape) with a shape as in Figure 2. Probably they have not any experience as in Figure 2.

In order to show pupils' thought processes, a conversation extract is given below as a proof. This consists of the conversation itself, descriptions of classroom context (given in italics) and analytic commentary (given in the right hand column).

Nur: Let's take 21.	They start with 25 and 21
Emrullah: Take this 22. There is also 22. *(while looking for the stick).*	Now trying to find third side. Emrullah chose nearest length.
Nur: Let's take a small one.	Nur suggests a small one.
Ayse: 22 is here. *(while showing the stick)*	
Caglar: *(takes 22 from Ayse)*	Caglar behaves as a group leader prefers Emrullah's choice. He prefers nearest length.
Ayse: this is a right angled triangle. *(it is not a right triangle. She said this without making a right triangle*).	She remembers a concept related with triangle. But she is not using this concept in the right place.
Nur: let's make a triangle at that time. *(At the same time they are trying to make a triangle with 21, 25 and 22)*	
Emrullah: it doesn't fit into these. It is too big. *(22 does not fit in the others. They did not change the positions of the sticks).*	But without changing their first attempt and without changing the angles in order to fit in their sides, they did not manage to obtain a triangle.
Caglar: *(changing the third stick with)*. 12, 12 is small *(while trying)*. Where is 13 *(find and try 13)*. *(The pupils did not change the position of the first two sticks)* Caglar: it did not form a triangle.	And they start to change the third side in order to make a triangle without changing the first position of the first two sides. Even they choose slightly small one than the first two they do not manage to make a triangle.
(They try 20. Nur changes the first position of the sides in order to fits them according to 20).	At the end they make a triangle by changing the position of the sides and using nearest lengths. (21, 25 and 20).
Nur: 20 exactly fits. Does it form a right angled triangle? *(It is difficult to decide this by just looking)*.	She is trying to fit the obtained triangle into her triangle image. The image is related to the right angled triangle.
Emrullah: it formed.	
(They are choosing the sticks in order	

to make another one).	
Nur: let's take 10. *(she puts it near the sticks which have been placed before: 13 and 10).*	Again they are making another one.
Caglar: *(brings 9).*	
Nur: it does not form a triangle.	
Caglar: how do you know?	
Nur: it forms a triangle.	
Caglar: not.	
Researcher: how do you decide this	
Caglar: because they are not touching *(he points the sticks).*	Same mistake. He knows a triangle a closed shape.
Nur: these must be equal. *(while pointing two sides).* *(She is replying the researcher question).*	She has other images in her mind. Some concepts related to triangles affect her hypothesis. Probably she remembers isosceles triangles. She knows this is a triangle.

(Group 2, from videotape)

In this conversation extract, two cases can be taken into account, Nur and Caglar.

Nur (a girl) has some isolated models. She is at the second stage of Hejny's model. In other words, she has some experiences and pieces knowledge about the concept of triangle and these are stored as isolated images. She has experiences related to triangles and she remembers the images of the right-angled triangle and the isosceles one. She examines whether their trials fit these images. Even as she makes a triangle but not an isosceles one, she decides this one does not form a triangle. Additionally, in some their trials, they do not manage to form a triangle by touching the end points of the sticks. But according to their choices, it must form a triangle. At this time Nur gives the reason why it did not form a triangle - its two sides are not equal.

Caglar (a boy) has some isolated models like Nur. At the beginning of the activity, he is at the second stage according to Hejny's model. But it seems that he is one step ahead according to Nur. Caglar does not reflect his thought process during the conversation. Caglar has an image of a triangle. Based on this image it seems that he has a hypothesis about how to make a triangle by using the sticks. Because he makes the final decision each time about which sticks have to be chosen. His friends recommend some of them but he decides which one fits in their trials.

In the following conversation extract, it is possible to see pupils' generalisation process.

Researcher: *(by pointing two sides)* Must these be equal? This and this one? If you make a different one, does it not form a triangle?	
Nur: At this time its base does not fit. *(she means that a base must horizontal according to her position).*	Because of the meaning of a base in her mind, she has difficulties to make a triangle.
Caglar: it does not form.	
Researcher: Why do you say that it does not form?	Researcher tries to understand the pupil's mind.
Nur: because these right sides (*by pointing at two sides*) are not equivalent.	
Researcher: which sides must be equivalent? Could you show me?	
Caglar: look this is a base and long.	His concept of base is the same as the longest side. He tries to make a hypothesis related to the base.
Researcher: Is there any other way to form one by using these? (*They are still discussing 10, 13 and 9*).	
Caglar: This is a little bit long (his *attention is still on the base*).	
Nur: Look at that (*she makes a triangle by changing the sticks' positions*)	
Researcher: that's great.	
Researcher: Why does it form? What is the difference between this and the trials before?	
Caglar: before, we tried to make a right angled triangle.	His thought in making a triangle is shaped by the image of right angled triangle.
Emrullah: let's make another one. This time we choose the small ones.	
(*They are making another one*).	
Caglar: 5, 6 (*takes 11*)	
Nur: this does not form a triangle.	

Researcher: Try and then decide.	
Nur: it does not form. Let's narrow them (*by referring to the angles*).	
Researcher: why does it form?	
Caglar: look, these two (*by pointing at two sides*) must be longer than the base.	Instantly, he says his hypothesis. This hypothesis is related to base and the other sides' lengths. In his mind, the base means the longest side.
Caglar: look, put 16 down. If both of the sticks are bigger than 16, they form a triangle (*he finds 10 and 15 and makes the triangl*e).	He is showing and at the same time testing his hypothesis.

(Group 2, from videotape)

After several trials, the isolated models do not appear to be so isolated. Caglar links these models to construct a new model. He produces a hypothesis; (this is what has been expected from pupils). His hypothesis is related the isolated models and a new model is constructed. This is the third stage according to Hejny's model, the stage of generalization. This means that he made connections between some isolated models.

Both of the pupils (Caglar and Nur) at the beginning of the activity (experiment) have fixed isolated models related to triangles. During the activity they try to complete the tasks based on these models. Because of the models are isolated, they have difficulties in completing the tasks. It takes time to reach the generalization required from the activity. When the pupils make the connections between isolated models, they start to produce new models. This experiment allows pupils to make connections between the isolated models.

It is observed that the pupils have difficulties in absolute value and subtraction of integers. These difficulties prevent pupils from seeing basic relations. Furthermore there are also pupils who can derive the generalization although they make operational mistakes. Some pupils find out the relationship among the side lengths of a triangle in the first worksheet. The reason why the pupils catch the critical points via strategic measurements while making the triangles is that they are exposed to this type of study.

The pupils come to the conclusions after several trial hypotheses on the worksheet 4.

4. CONCLUSION

This experiment allows the pupils not only to use basic concepts related to triangle but also help them to form the triangle concept.

This study shows that the pupils reached generalizations despite their initial wrong and deficient knowledge of models. During the tasks, by using sticks the pupils make trials then develop some hypothesis. This experiment allows the pupils to test and develop their hypothesis despite their wrong and weak basis for the concept.

Additionally, these activities not only allow the pupils to construct new knowledge but also to correct and change old ones. As Tall (1991) said, the existing schema needs to change when it is inadequate to assimilate new knowledge (cited in Dahl, 2004: 136).

To sum up, the pupils with prior experience that is stored as isolated images and wrong concept knowledge have difficulties in constructing new knowledge. But activities such as those designed for this study give a chance for pupils to examine their thoughts by practical work (testing), it allows pupils to make links with the prior experience and construct new images (piece of knowledge).

It is very important to establish how the pupils understand the mathematical concepts. When the aforementioned subject is traced, the accuracy of the old knowledge is also tested. Teachers may perform more effective mathematics teaching by considering thought processes in developing concepts.

In order to extend this study, it is possible to develop another activity which asks for the unknown side of a triangle. For example two lengths can be given and the pupils asked to find a third length, the three lengths forming a triangle. The present study can be expanded to quadrilaterals as well.

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